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Probabilistic Structural Analysis Methods (PSAM) for Select Space Propulsion System Components – II (6th Annual Report)

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1 PROJECT OVERVIEW

1.1 Introduction

This report summarizes the FY'90 technical developments of the NESSUS system for component risk assessment for the Probabilistic Structural Analysis Methods for Select Space Propulsion Components contract. The technical efforts focused on a new automated reliability algorithm, an extension of the existing advanced mean value probabilistic algorithm, resistance models which include state-of-the-art models, multi-factor interaction model, and a probabilistic constitutive relation model, and a risk module that can compute the risk with respect to cost, performance, and a user-defined criteria.

The team for the FY'90 effort consisted of the following individuals and organizations:

SwRI: Dr. T.A. Cruse
Mr. H.R. Millwater
Mr. B.H. Thacker
Dr. S.V. Harren
Dr. Y.-T. Wu
Dr. Y. Tornig
Dr. R. Aithal
Ms. J.P. Buckingham

Rocketdyne: Dr. K.R. Rajagopal

University of Arizona: Prof. P.H. Wirsching

1.2 Summary of FY'90 Accomplishments

The principal focus of the component reliability subtask was the development and implementation of a new reliability algorithm. This algorithm is described in Chapter 2. In addition to a new reliability algorithm, the Advanced Mean Value algorithm present in NESSUS 4.2 was automated. This feature is also discussed in Chapter 2.

The component resistance subtask focused on two efforts, NESSUS coding and resistance model formulation. The NESSUS code was enhanced so that very general resistance models, even those which involve nonlinear combinations of finite element results and material resistance variables, can be analyzed in a straightforward manner. The new capabilities are documented in Chapter 3.

A number of resistance models are formulated according to the statement-of-work by SwRI, Rocketdyne, and the University of Arizona. A summary of the investigated and implemented models is given in Chapter 3.

The component risk subtask focused on the development and implementation into NESSUS of algorithms to compute the structural risk. The risk with respect to cost, performance, and a user-defined criteria can be computed by NESSUS. This capability is discussed in Chapter 4.

Various supporting technology was developed in FY'90 in support of component reliability, resistance, and risk, and is discussed in Chapter 5. A fast convolution method for FPI was developed. A probabilistic interface was developed around the BEST3D boundary element program and the expert system help screens were enhanced.

The above-mentioned technology is contained in NESSUS 5.0. The release notes for NESSUS 5.0 are contained in Appendix A.

1.3 Future Effort

The FY'91 effort will focus on system risk assessment. The enhancements to NESSUS in FY'90 for component reliability, resistance, risk, and supporting technology lead directly into the FY'91 work on system reliability.

1.4 Publications

The following papers were presented and/or published during FY'90.

1. "Application of Probabilistic Structural Modeling to Elastoplastic and Transient Analysis," T.A. Cruse, H.R. Millwater, S.V. Harren, and J.B. Dias, presented at IFIP WG 7.5 Working Conference, Berkeley, California, March 26-28, 1990 (Proceedings to be published by Springer-Verlag).
2. "Application of the Probabilistic Approximate Analysis Method to a Turbopump Blade Analysis," B.H. Thacker, R.C. McClung, and H.R. Millwater, AIAA/ASME/ASCE/AHS/ASC 31st Structures Structural Dynamics and Materials Conference, Long Beach, California, 2-4 April 1990.
3. "Computational Methods for Probability of Instability Calculations," Y.-T. Wu and O.H. Burnside, AIAA/ASME/ASCE/AHS/ASC 31st Structures, Structural Dynamics and Materials Conference, Long Beach, California, 2-4 April 1990.
4. "Probabilistic Analysis of a Materially Nonlinear Structure," H.R. Millwater, Y.-T. Wu, and A.F. Fossum, AIAA/ASME/ASCE/AHS/ASC 31st Structures, Structural Dynamics and Materials Conference, Long Beach, California, 2-4 April 1990.
5. "Probability Approach for Strength Calculations," C.C. Chamis and T.A. Cruse, AGARD Structures and Materials Panel Workshop, 70th SMP Meeting, Sorrento, Italy, April 2-6, 1990.
6. "Probabilistic Structural Analysis, Reliability and Risk of Critical SSME Components," T.A. Cruse, C.C. Chamis, and K.R. Rajagopal, presented at 1990 Conference on Advanced Earth-to-Orbit Propulsion Technology, Huntsville, Alabama, May 15-17, 1990.
7. "Structural Reliability and Resistance Modeling with the NESSUS Software System," S.V. Harren, H.R. Millwater, and B.H. Thacker, presented at the 26th AIAA/SAE/ASME/ASEE Joint Propulsion Conference, Orlando, Florida, July 16-18, 1990.
8. "Probabilistic Structural Analysis Methodology and Applications to Advanced Space Propulsion System Components," T.A. Cruse, K.R. Rajagopal, and J.B. Dias, presented at the Symposium on Computational Technology for Flight Vehicles, Washington, D.C., November 5-7, 1990.

2 COMPONENT RELIABILITY

This task focused on developing and implementing the reliability algorithms in NESSUS. The existing advanced mean value algorithm (AMV) p -level procedure was enhanced, and a new AMV based z -level procedure was developed. The p -level algorithm is used to compute the response value corresponding to a specified probability. The p -level procedure is most useful when computing the entire CDF. The z -level algorithm is used to compute the probability corresponding to a specified response, i.e., the structural reliability.

2.1 AMV+ p -level Procedure

The AMV+ p -level procedure is used to compute the response corresponding to a specified probability, or p -level. In NESSUS, the method is used when points along the entire range of the CDF are to be computed or the response at specified probability levels is to be computed. This procedure is selected by setting *ANALYTYPE = 0 or 2 in the FPI input section of PFEM.

Table 2.1 gives a schematic of the method, along with the number of finite element solutions that are required to perform a first order probabilistic analysis. In Table 2.1, N is the number of input random variables and M is the number of p -levels. MVFO refers to "mean value first order," and AMVFO refers to "advanced" MVFO. In an MVFO analysis, $(N+1)$ sensitivities are computed and used to construct a linear performance function (g -function) about the mean values of the random variables. In step 2, an update (or "move") is performed, which entails a finite element solution, to update the response value at each of the M probability levels.

A fully automated AMV+ p -level algorithm has been implemented in NESSUS. The user inputs an allowable number of iterations and a convergence tolerance. At each p -level, NESSUS continues iteration until either: *i*) the allowable number of iterations is reached, or *ii*) the relative change in z is within the convergence tolerance.

2.2 AMV+ z -level Procedure

The AMV+ z -level procedure is used to compute the probability corresponding to a specified response or z -level. In NESSUS, the method is primarily used for reliability calculations, where p must be calculated for a specific value of z .

The key to efficient reliability analysis is the ability to compute quickly the location of the most probable point (MPP). To reduce the number of finite element solutions required while searching for the MPP, an algorithm has been devised and implemented to estimate a good starting MPP. Once the estimate is obtained, a straightforward iteration procedure is used to converge on the desired MPP. It should be noted that the sole purpose of the algorithm used to obtain the starting MPP is to minimize the number of finite element solutions, and can be modified or adapted as experience warrants.

The AMV+ z -level procedure works in " u -space," which is related to p by $2p = 1 + \text{erf}(u/\sqrt{2})$, and "erf" is the error function. An outline of the AMV+ z -level procedure is given below.

A. Compute an initial estimate for the most probable point.

1. Estimate the probability level u using $u = az^2 + bx + c$, where a , b , and c are determined from:
 - i*) u at the mean value of z ,
 - ii*) $\partial u / \partial z$ at the mean value of z , and

- iii) the AMV result (move) at $u = +5$ or -5 , depending on whether the z -level is greater than or less than its median, respectively.
 - 2. From the probability level u computed in Step A-1, compute the MPP using FPI and the response using a move (AMV procedure).
 - 3. Compute a new estimate of the probability level u using $u = az^2 + bx + c$, where a , b , and c are determined from:
 - i) u at the mean value of z ,
 - ii) u at the MPP obtained in Step A-2, and
 - iii) the AMV result (move) at $u = +5$ or -5 , depending on whether the z -level is greater than or less than its median, respectively.
 - 4. From the probability level u computed in Step A-3, compute the starting MPP using FPI.
- B. Given a starting u and MPP:
 - i) obtain sensitivities at the MPP, and
 - ii) estimate an improved u and MPP for the input z -level using FPI.
- C. Repeat Step B until either:
 - i the allowable number of iterations is reached or
 - ii the relative change in u is less than the convergence tolerance.

A significant ingredient of these two algorithms is that NESSUS will automatically iterate until either a maximum number of iterations is reached or a user-specified convergence tolerance is satisfied. For the p -level algorithm, the convergence is based on the computed z between two successive iteration steps, i.e., convergence is achieved when $|(z_i - z_{i-1})/z_i| < \text{tolerance}$. For the z -level algorithm, convergence is achieved when $|(u_i - u_{i-1})/u_i| < \text{tolerance}$. With this procedure even highly nonlinear problems can be analyzed with confidence.

As an example, consider a thick cylinder under internal pressure with a perfectly plastic material model. This problem was shown to have a nonlinear response in the paper presented at the 31st SDM conference, "Probabilistic Structural Analysis of a Materially Nonlinear Structure," by H. Millwater, et al. The computed CDF using the AMV+ p -level algorithm is shown in Figure 2.1. Because of the nonlinear response function, the AMV move was not sufficiently accurate, 1st iteration was necessary. However, in this problem, an analytical solution was known which will not be known in general. Thus, in general, the engineer will not know when the solution is sufficiently accurate. With the new iteration capability, NESSUS will automatically iterate until convergence. Figure 2.2 shows the comparison of the converged NESSUS solution with Monte Carlo. Figure 2.3 shows the probabilistic sensitivity factors for this problem. NESSUS iterated until convergence occurred after the second step of iteration 1.

The z -level algorithm was also used to analyze this problem. Table 2.2 shows the results for several iterations at several z values. The computed probability at each z level corresponds with the p -level results.

These algorithms have also been exercised thoroughly in the validation problems located in Appendix C.

Table 2.1
Schematic of AMV+p-levels Procedure

Iteration	Step	Analysis	Number of Finite Element Solutions
0	1	MVFO	$N+1$
0	2	AMVFO (Move)	$+M$
1	1	First Order	$+(N+1) \times M$
1	2	Move	$+M$
2	1	First Order	$+(N+1) \times M$
2	2	Move	$+M$
.	.	.	.
.	.	.	.
.	.	.	.

Table 2.2
Elastic-Plastic Cylinder

PFEM z-level Validation Problem

z-level	U	mvfo	quad est 1	quad est 2	1st iter	2nd iter	3 iter	5 iter
-17581	-11.2646	-0.4841E+01	-0.5744E+01	-0.4958E+01	-0.4957E+01	-0.4957E+01	-0.4957E+01	-0.4957E+01
-15825.	-8.6754	-0.4960E+01	-0.4765E+01	-0.3998E+01	-0.3997E+01	-0.3997E+01	-0.3997E+01	-0.3997E+01
-14018.	-6.0063	-0.4274E+01	-0.3572E+01	-0.3031E+01	-0.3033E+01	-0.3033E+01	-0.3033E+01	-0.3033E+01
-12273.	-3.3547	-0.2834E+01	-0.2351E+01	-0.2117E+01	-0.2058E+01	-0.2058E+01	-0.2058E+01	-0.2058E+01
-10897.	-1.2257	-0.1161E+01	-0.1120E+01	-0.1081E+01	-0.1071E+01	-0.1067E+01	-0.1067E+01	-0.1067E+01
-9573.8	0.8713	0.9035E+00	0.9496E+00	0.9259E+00	0.9308E+00	0.9304E+00	0.9304E+00	0.9304E+00
-8031.3	0.7340	0.1896E+01	0.2025E+01	0.1941E+01	0.1941E+01	0.1941E+01	0.1941E+01	0.1941E+01
-6524.6	0.5474	0.2903E+01	0.2903E+01	0.2955E+01	0.2955E+01	0.2955E+01	0.2955E+01	0.2955E+01
-8016.0	0.3640	0.3992E+01	0.4103E+01	0.3971E+01	0.3971E+01	0.3971E+01	0.3971E+01	0.3971E+01

* Limited to U = -5

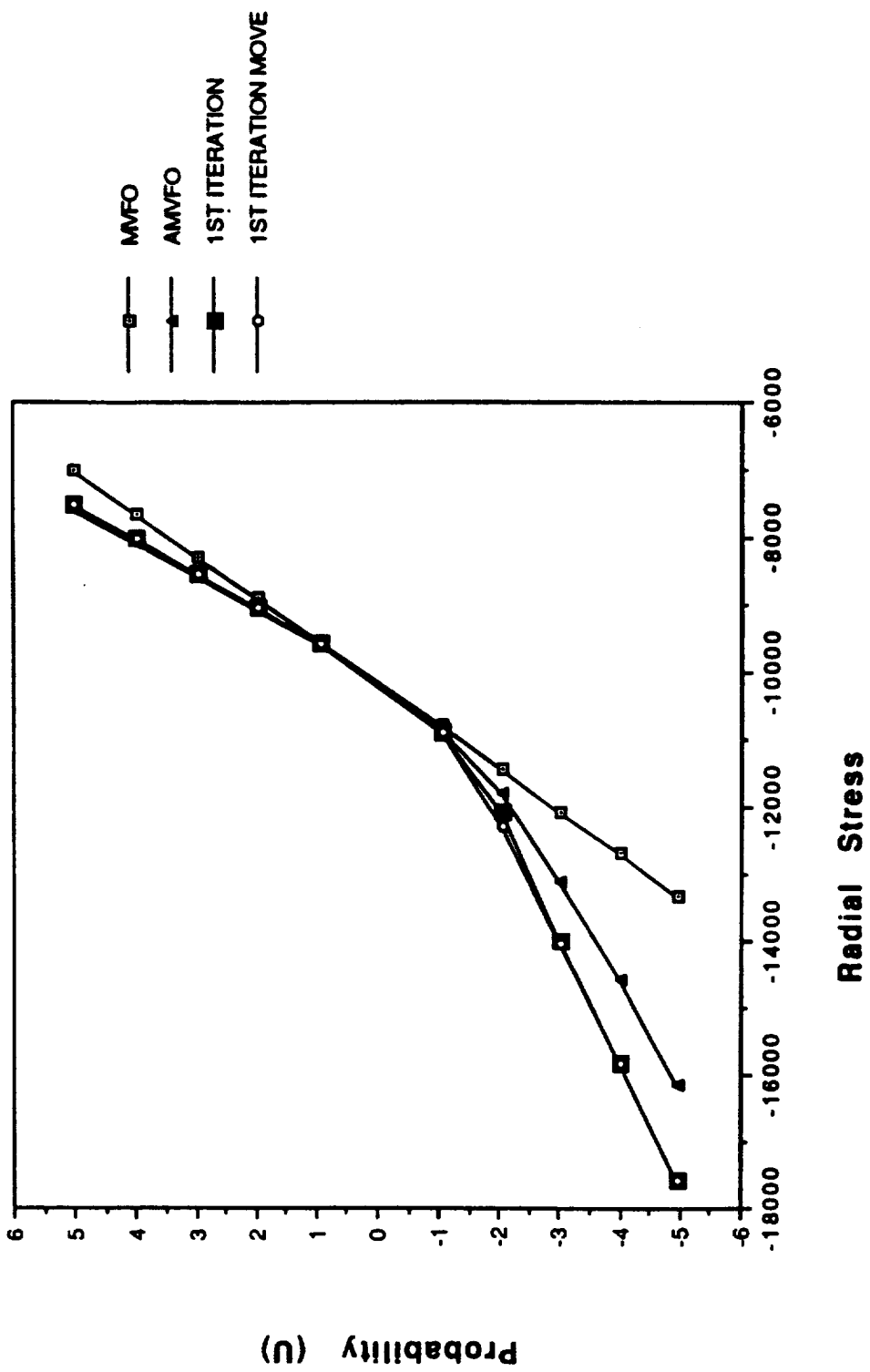


Figure 2.1 NESSUS 5.0 Iteration Algorithm Applied to Elastoplastic Cylinder

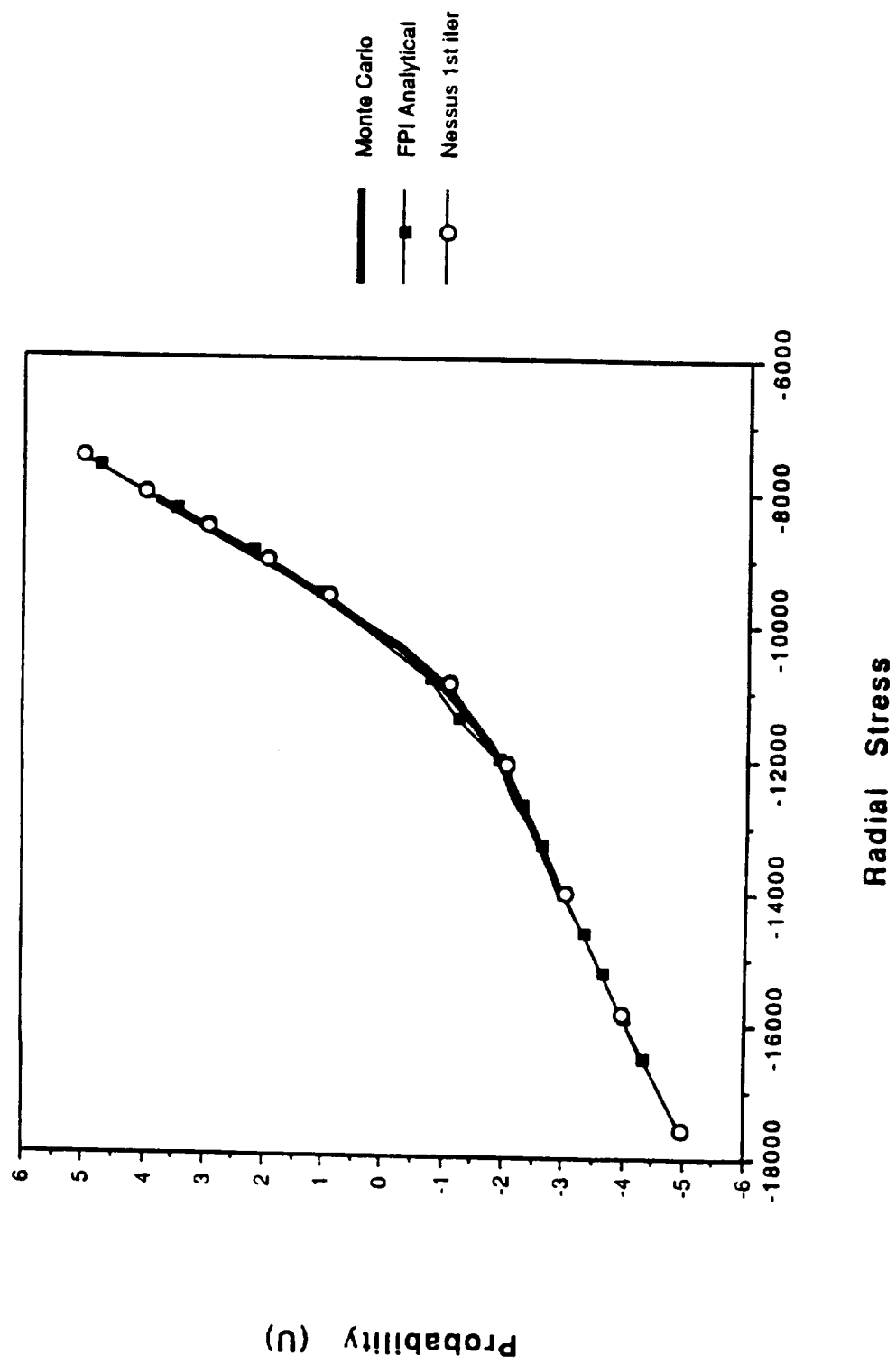


Figure 2.2 NESSUS 1st Iteration Results Compared with Monte Carlo and FPI Analytical

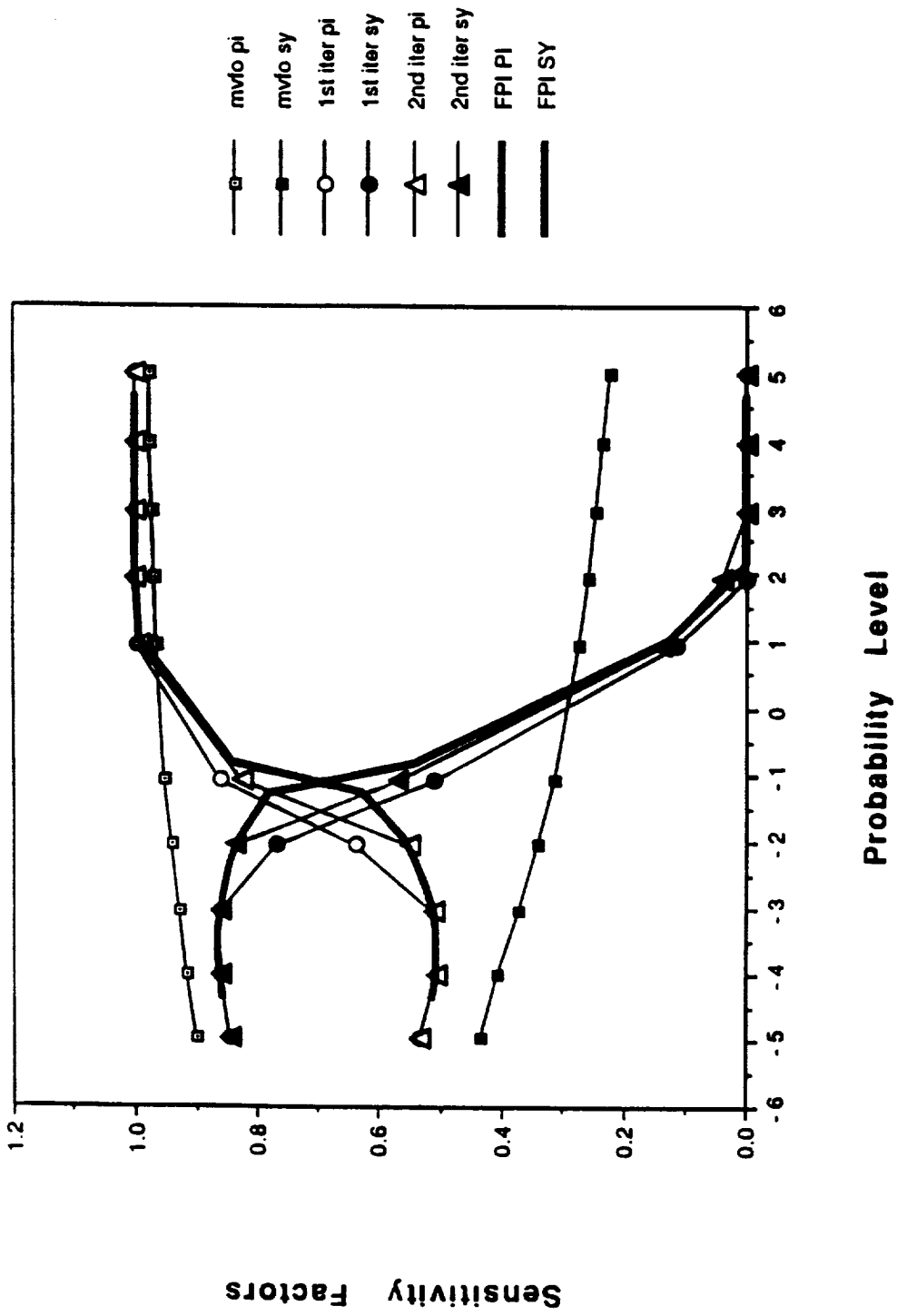


Figure 2.3 Sensitivity Comparison of AMV+ Algorithm with FPI on Analytical Equations

3 COMPONENT RESISTANCE

3.1 Introduction

This section presents the work accomplished under Task V-B "Component Probabilistic Structural Resistance." The objective of Task V-B is to adapt/modify the computer codes (NESSUS) to simulate computationally the uncertainties in component structural resistance.

3.2 Status

Structural resistance modeling has been fully integrated in the NESSUS computer program. This capability was achieved through several key developments:

- A completely general facility for modeling structural resistances was integrated within the framework developed under Task V-A for computing structural reliability.
- A new data processor was developed and integrated into NESSUS that allows NESSUS/FEM results such as stresses, strains, and displacements to be used in defining other response measures such as maximum principal stress, plastic strain range, or RMS displacement.
- A new random variable type "COEF" (for coefficient) was added to NESSUS/FEM. This new random variable provides the capability for modeling probabilistic material damage, which is defined in terms of some number of COEF random variables.
- Whenever possible, options were added to allow the user to provide custom resistance models, post-processing options, etc., via user-programmable subroutines.
- The input format and reader were completely rewritten to simplify and streamline the additional input required.

The following sections provide a more detailed description of the work completed under this task.

The overall approach to resistance modeling in NESSUS is described in Section 3.3, followed by a summary of the required code structure changes in Section 3.4. A significant amount of research was performed to determine what types of resistance models should and could be incorporated within the computational framework of NESSUS. From this work, a database of resistance (or reliability) models was compiled. In Section 3.5.1, this database is categorized and described. Models based on simple limiting values of structural response are described in Section 3.5.1.1. Models based on material strength are described in Section 3.5.1.2, and models that attempt to account for material degradation due to damage are described in Section 3.5.1.3. Next, in Section 3.5.2, the multi-factor interaction (MFI) model is described, and a formulation for the probabilistic modeling of an austenitic stainless steel is presented in Section 3.5.3. A model for defining a random stress-strain curve in NESSUS/FEM is described in Section 3.6, and several demonstration problems are presented in Section 3.7.

3.3 Approach to Structural Reliability Computation Using NESSUS

The reliability of a structure can be viewed as a measure of the ability of the structure to perform as designed. However, because uncertainties in the design process are unavoidable, a probabilistic approach to estimating the chance of non-performance is needed. Probably the simplest statement of performance for a structure is that the available strength of the structure be adequate to withstand the maximum loads acting on the structure. This performance can be stated mathematically [1] as

$$p_s = P(S - R \leq 0) \quad (3.1)$$

where p_s is the probability of safety (reliability), S is the loading or "stress" variable associated with the structure, and R is the strength or "resistance" measure of the structure. In general, most structures will be required to meet more complex performance measures.

The probabilistic analysis methods used in NESSUS are based on the concept of a limit-state [2]. A limit-state function $g(X)$ is formulated in terms of the input random (i.e., engineering) variables, X . The limit-state function, also referred to as the performance function, is written such that $g(X) = 0$ defines the boundary between the "failure" and "safe" regions. In other words, $g(X) \leq 0$ indicates failure and $g(X) > 0$ indicates safety.

To describe the new approach, it is useful to review the methodology used in the earlier versions of NESSUS (i.e., < 4.8). Previously, a structural response function $Z = Z(X)$ was defined either by a numerical method such as NESSUS/FEM or by a closed-form expression. In a probabilistic structural response analysis, the performance function was formulated as $g(X) = Z(X) - Z_0 = 0$, where Z_0 can be thought of as a limiting value of Z . Point probability estimates are made using specific limits for Z_0 and the cumulative distribution function (CDF) is obtained by varying Z_0 .

The approach for defining more general performance functions in NESSUS is straightforward. Instead of interpreting $Z = Z(X)$ as being solely the structural response, its definition is extended to include the effects of both the response and material/resistance model. Specifically, $Z(X)$ is formulated to represent the *overall* response function. In a component structural reliability analysis, $Z(X)$ will be a function to predict a particular failure mode. Note that the probabilistic structural response analysis available in previous versions of NESSUS is a subset of this approach.

3.4 Code Structure

NESSUS/PFEM, as in past versions of NESSUS, is the main driver routine in NESSUS 5.0. The input format for PFEM in NESSUS 5.0 has been completely rewritten to be more organized and user-friendly. In particular, all input is now column and case independent and is completely keyword driven. A number of new keywords have been added reflecting new code capabilities. Appendix A gives a detailed presentation of the NESSUS 5.0 code structure.

3.5 Combined Stress and Resistance Modeling

Material/resistance modeling capabilities have been researched and implemented in the NESSUS Software System and are grouped as follows: (1) state-of-the-art (SOA) models, (2) a multi-factor interaction relation, and (3) a probabilistic constitutive relationship. All of the models are integrated with the automated Advanced Mean Value Iteration (AMV+) algorithms described in Chapter 2.

Resistance Model Research

Measures of structural resistance vary from analysis to analysis. Consequently, it would be inefficient to implement a large number of resistance models. The approach taken in this task was to (1) develop a code structure that offered the flexibility of defining resistance models through user-programmable routines and (2) include a broad cross-section of "pre-coded" widely used resistance models that could be used both for design purposes and as examples for the engineer to use in defining more specific resistance models.

A considerable portion of time was devoted to surveying the different types of resistance measures that would be most applicable to the space propulsion industry needs. Southwest Research Institute, The University of Arizona, and Rocketdyne all participated in this research. By performing this research early in the fiscal year, the types of data were identified, and were subsequently used in the design of the new PFEM code.

A database of resistance models was compiled as a result of an extensive review of classical resistance measures (University of Arizona Report, Appendix B.1) and resistance models currently being used in the liquid rocket engine industry (Rocketdyne Report, Appendix B.2). A summary table of the more commonly used measures of resistance is given in Table 3.1. The reader is referred to the respective appendix for complete details.

The models listed in Appendix B.1 are grouped into the following categories:

- Stress-Based Fatigue Life Prediction Models
- Strain-Based Fatigue Life Prediction Models
- Fracture Mechanics Crack Growth Models
- High Temperature Low Cycle Fatigue Models
- Linear Elastic Fracture Mechanics Resistance Models
- Fatigue and Creep Resistance Models

Within each category, a number of models are presented along with a discussion of modeling approaches, statistical input considerations, and reliability analysis recommendations. As a result, it should be a straightforward task to incorporate a model, or some variation, from this database into the NESSUS 5.0 user-programmable routines.

3.5.1 State-of-the-Art Models

The NESSUS state-of-the-art material/resistance models are categorized into three groups: (i) design factor models, (ii) material strength models, and (iii) material degradation models.

Table 3.1
Summary of SOA Resistance Models

No.	Name and/or Example SOA	Data Description/Origin	CDF	Probability	Considerations
1	Displacement	user: δ_0 FEM/user: δ	δ	$P[\delta \leq \delta_0]$	• Data Combinations • Data Transformations
2	Frequency	user: f_0 FEM/user: f	f	$P[f \leq f_0]$	• Min/Max limits • $f_0 = f_0(n\Omega)$
3	Buckling load	user: λ_0 FEM/user: λ	λ	$P[\lambda \leq \lambda_0]$	• Combinations/Transformations
4	Stress	user: σ_0 FEM/user: σ	σ	$P[\sigma \leq \sigma_0]$	• Combinations/Transformations
6a	Fracture Mechanics- (Limit toughness) Fatigue life: for $n \neq 2$ $N_f = \frac{2[a_f^{(1-n)/2} - a_0^{(1-n)/2}]}{C(2-n)(Y\sigma_{max}\sqrt{\pi})^n}$ for $n = 2$ $N_f = \frac{\ln(a_f/a_0)}{C(Y\sigma_{max}\sqrt{\pi})^2}$ where $a_f = \frac{1}{\pi} \left(\frac{K_{Ic}}{Y\sigma_{max}} \right)^2$; valid for $\sigma_{max} = \Delta\sigma$	user: K_{Ic} user: C user: n user: $f \neq f(a)$ user: a_0 user: αN_{max} ; $0 < \alpha < 1$ FEM/user: σ_{max}	N_f	$P[N_f \leq \alpha N_{max}]$	• Combinations/Transformations • Different stress measures

6b	Very LCF (Coffin-Manson): $N_f = A(\Delta\epsilon_p)^b$	user: \bar{A} user: b user: αN_{mean} ; $0 < \alpha < 1$ FEM/user: $\Delta\bar{\epsilon}_p$	N_f	$P\{N_f \leq \alpha N_{\text{mean}}\}$	<ul style="list-style-type: none"> Combinations/Transformations Different strain measures
6c	LCF: $N_f = A 10^{C\sigma_{\text{mean}}}(\Delta\epsilon)^b$ where $\sigma_{\text{mean}} = \frac{\sigma_{\text{max}}}{2}$ if $\sigma_{\text{min}} = 0$	user: \bar{A} user: b user: C user: αN_{mean} ; $0 < \alpha < 1$ FEM/user: $\bar{\sigma}_{\text{mean}}$ FEM/user: $\Delta\bar{\epsilon}$	N_f	$P\{N_f \leq \alpha N_{\text{mean}}\}$	<ul style="list-style-type: none"> Combinations/Transformations Different stress/strain measures
6d	TMF: $N_f = A(\Delta\epsilon)^b$ (Separate out bulk thermal strain)	user: \bar{A} user: b user: αN_{mean} ; $0 < \alpha < 1$ FEM/user: $\Delta\bar{\epsilon}$	N_f	$P\{N_f \leq \alpha N_{\text{mean}}\}$	<ul style="list-style-type: none"> Combinations/Transformations Different strain measures
6e	HCF: $N_f = A(\sigma'_s)^b$ $\sigma'_s = f(\sigma_s, \sigma_m, \sigma_a)$	user: \bar{A} user: b user: αN_{mean} ; $0 < \alpha < 1$ FEM/user: $\bar{\sigma}'_s$	N_f	$P\{N_f \leq \alpha N_{\text{mean}}\}$	<ul style="list-style-type: none"> Different stress measures, σ'_s (Goodman, Gerber, Soderberg)
6f	HCF-allowable operating stress amplitude	user: $\beta \bar{\sigma}_{\text{mean}}$; $0 < \beta < 1$ FEM/user: $\bar{\sigma}'_s$	σ'_s	$P\{\sigma'_s \leq \beta \sigma_{\text{mean}}\}$	<ul style="list-style-type: none"> Different stress measures (see 6c)
7a	Creep-strain limit	user: $\bar{\epsilon}'_f$ FEM/user: $\bar{\epsilon}$	ϵ'	$P\{\epsilon' \leq \bar{\epsilon}'_f\}$	

7b	Time to Rupture: (Larson-Miller) $t_r = 10^{P(\log(T - T_0) + C)}$	user: \dot{C} user: T_A user: f_0 user: Larson-Miller plot $P(\sigma)$ FEM/user: $\dot{\epsilon}$ FEM/user: $\dot{\sigma}$	t_r	$P[t_r \leq t_0]$	• Different Stress Measures
8.	Ratcheting (cycles to total accumulated ratchet) $r = r(N)$ where eg. $r = \epsilon'$	user: N_{lim} user: f_{lim} FEM/user: nN table (compute N_r from $r_{lim} = r(N_r)$)	N_r	$P[N_r \leq N_{lim}]$	• Uncertain nN curve
9.	Material degradation: (Time @ temperature damage) $D = \sum_{i=1}^n \frac{t_i}{t_{Li}}$ $t_{Li} = A \cdot \exp\left\{\frac{\Delta H}{RT_i}\right\}$	user: \dot{D}_{lim} user: $\dot{\lambda}$ user: R user: $\Delta \dot{H}$, $[\Delta H \neq \Delta H(\sigma)]$ user: time t_i at temp T_i for $i = 1, \dots, n$	D	$P[D \leq D_{lim}]$	

3.5.1.1 Design Factor Models

Design factor models are traditional structural design factors, or more simply, limiting conditions. (A probabilistic analysis using a *non-random* design factor would be termed a probabilistic structural response analysis, described earlier in Section 1.3.) Example design factors include deflections, natural frequencies, and stresses. In NESSUS 5.0, design factors themselves can be random functions, and could be thought of as the resistance curve.

3.5.1.2 Material Strength Models

Material strength models model the uncertainty associated with the ability of the material to "resist" its loading. Typical material strength measures include yield stress, fracture toughness, and elongation.

3.5.1.3 Material Damage Models

Material damage models attempt to consider the effects of cumulative material damage and may or may not affect the structural response, the choice being dependent on the particular model used. Examples of material damage models include low cycle and thermomechanical fatigue, crack growth, and stress corrosion.

3.5.2 Multi-Factor Interaction Relation

The Multi-Factor Interaction (MFI) relation [3,4] models material degradation by using a series product equation fitted to experimental or assumed data. A single term is used to describe each primitive variable (stress, temperature, cycles, etc.). The general form of the MFI relation is

$$\frac{M_p}{M_{p_0}} = \prod_{i=1}^n \left(\frac{A_{ip} - A_i}{A_{ip} - A_{i0}} \right)^{a_i} \quad (3.2)$$

where A_{ip} , A_i , and A_{i0} are the ultimate, current, and reference values of a primitive variable, a_i is the value of an empirical constant for the i^{th} primitive variable, n is the number of primitive variables considered, and M_p and M_{p_0} are the current and reference values of the material property. As seen, the effect of each primitive variable on the current material property varies from 1.0 (no effect on strength) to 0.0 (complete loss of strength) times the reference value of the material property.

3.5.3 Probabilistic Constitutive Relation

The goal of this research was to develop simplified relations to describe probabilistic material behavior in terms of micromechanical parameters. Although this approach is more difficult to formulate than the macromechanical formulations discussed earlier and is material dependent, it can provide valuable information regarding what drives basic randomness in material properties and behavior.

A formulation for the probabilistic behavior of a polycrystalline austenitic stainless steel has been formulated. A complete description is given in Appendix B.3. In this presentation, the modeling and behavior of the austenitic stainless steel is broken down into (1) uniaxial stress-strain behavior, (2) low cycle fatigue response, (3) high cycle fatigue response, (4) crack growth behavior, and (5) creep rupture behavior. Within each category, the basic behavior is developed in terms of micromechanical parameters such as grain size, constituents, and porosity. Recommendations are also made for the selection of random variables.

3.6 Random Stress-Strain Curve Model

An important capability was implemented in NESSUS for modeling the scatter found in most stress-strain curve data. The stress-strain curve model is expressed as a function of seven (random) engineering parameters: Young's modulus, yield stress, initial plastic hardening slope, ultimate stress, strain at ultimate, Poisson's ratio, and a parameter governing "mixed" hardening. These parameters are in general correlated and non-normally distributed. A method has been formulated and coded into NESSUS to perturb consistently the stress-strain curve inputs to NESSUS/FEM. A detailed discussion of this model along with an example problem is presented in Appendix B.4.

3.7 Demonstration Problems

To demonstrate the new capabilities offered in NESSUS 5.0, several demonstration problems are presented. Here, only a summary of each problem is given, with details given in Appendix C. The summary includes a discussion of the options used, which particular resistance model is employed, and any special features that are exercised. In all cases, the automated AMV+ algorithms are used. Moreover, each problem is executed to completion with a single computer run and input deck.

3.7.1 Displacement Response of a Tilted Beam

In this demonstration, the response is the displacement interference between the tip displacement δ of a beam, tilted at a 30° angle from the x-axis, and a displacement limit δ_0 , which has some uncertainty associated with it. This problem provides a demonstration of a simple design factor model (Section 3.7.1), where the end response of the beam is the "stress" variable, and the (uncertain) limiting response is the "resistance" variable. Another important aspect exercised in this problem is the transformation of displacements into another coordinate system during the course of the analysis. A complete write-up is provided in Appendix C.1.

3.7.2 Buckling Example

This example demonstrates the computation of structural reliability with respect to buckling limits. The solution procedure is similar to that described in Section 3.7.1. A complete write-up is provided in Appendix C.2.

3.7.3 Stress Response of a Circular Disk

Here, a circular disk is analyzed with two equal and opposite forces P acting along a diameter. Failure is assumed when the maximum compressive stress, σ , due to the loads exceeds some limiting stress σ_0 . Therefore, the probability of failure is given by $P_f = P[\sigma_0 \leq \sigma]$. Once P_f is computed, the reliability is computed as $1 - P_f$. This is a demonstration of a material strength resistance model (Section 3.5.1.2), where σ_0 is the uncertain failure or yield stress. A complete write-up is presented in Appendix C.3.

3.7.4 Fatigue Life of a Three Point Bend Specimen

In this demonstration, the reliability of a three point bend specimen against failure by fracture is computed. Failure is assumed when the computed number of cycles to failure N_f is less than the design life N_0 , where N_f is computed using a simple Paris relation to describe the crack growth. This demonstration problem provides an example of a material damage resistance model (Section 3.5.1.3).

Another important aspect demonstrated in this problem is the use of both "FEM" and "non-FEM" random variables, respectively termed "computational" and "explicit," in NESSUS. (See the NESSUS 5.0 Release Notes in Appendix A for a more detailed explanation.) Examples of computational random variables include the loading and beam dimensions. Examples of explicit random variables include the initial crack length and the Paris exponent. A complete write-up is presented in Appendix C.4.

3.7.5 Elastoplastic Low Cycle Fatigue Life of a Rectangular Plate

Here, the probabilistic life of a rectangular plate is analyzed considering damage caused by low cycle fatigue. The Multi-Factor Interaction (MFI) relation is used in the analysis to degrade the initial yield stress as a function of the thermal fatigue cycles and temperature. Important aspects in the demonstration include (1) the use of both computational and explicit random variables to construct the performance function (in this case, N_f), (2) the automatic processing of 50 increments of stress and plastic strain to compute mean stress and plastic strain range, (3) the use of COEFFICIENT random variables in NESSUS/FEM to track the random parameters in the MFI model, and (4) the inclusion of material damage during the course of the finite element calculations via the MFI relation. A complete write-up is presented in Appendix C.5.

3.8 References

1. Ang, A.H.-S. and Tang, W.H., Probability Concepts in Engineering Planning and Design, Vol. 2, Wiley, New York, 1984.
2. Wu, Y.-T., "*Fast Probability Integration (FPI) Theoretical Manual*," Southwest Research Institute Contract Report, NASA Contract NAS3-24389, 1989.
3. Boyce, L. and Chamis, C.C., "Probabilistic Constitutive Relationships for Cyclic Material Strength Models," *29th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference*, Williamsburg, VA, April 1988.
4. Boyce, L. and Chamis, C.C., "Probabilistic Constitutive Relationships for Material Strength Degradation Models," *Proc. 30th Structures, Structural Dynamics and Materials Conf.*, Mobile, AL, April, 1989.

4 COMPONENT RISK

A risk module has been added to NESSUS, called NESSUS/RISK, which will compute the risk with respect to cost, performance and a user-defined criteria. RISK can be run stand-alone or in conjunction with PFEM. Analysis with PFEM is fully automated from a single input file.

4.1 Cost Algorithm

Risk with respect to cost is computed using the formula

$$\text{Risk/Cost} = C_0(x) + P_f(x) * C(x) + (1-P_f) * NC(x)$$

where x is the allowable response in a reliability analysis such as stress, displacement, cycles to failure, C_0 is the initial cost as a function of x , P_f is the probability of failure as a function of x , and $NC(x)$ is the probability of non-failure as a function of x . The $P_f(x)$ curve may be the CDF computed by NESSUS. However, in general, x could be any design parameters including non-random variables.

The four functions, $P_f(x)$, $C_0(x)$, $C(x)$, and $NC(x)$, are defined with appropriate keywords defined below. The failure function, $P_f(x)$, is defined in table form using the *XPF keyword. The three cost functions, $C_0(x)$, $C(x)$, and $NC(x)$ are defined using the *CFUNC keyword. The XPF and CFUNC x values do not have to coincide. The output points, x_i , at which the risk is to be computed are defined with the *XCOUT keyword. The output points do not have to coincide with the failure function or cost function points. RISK will linearly interpolate the failure function and cost function values to use in the equation for risk. However, no extrapolation is allowed; therefore, the output points must lie within the failure function and cost function data.

Example:

The user has computed a P_f function which is the probability of failure of a structure as a function of a member area, A . Thus, x is the member area A . (Note, in this example, x is not a response.) The initial costs are assumed to vary linearly with A . The cost of a failure is assumed to be independent of A , thus, the cost function, is assumed to be constant with A . The cost of non-failure is also assumed to be constant with A although at a far less magnitude than the cost of failure function. The user desires the risk at a number of points between the areas 0.1 and 1.0.

The input file would look like:

```
*RISK          to run risk stand-alone, start file with *RISK
C              comments are allowed
C
C              define the probability of failure function
C              five points will be used
*XPF
  *TABL          5      indenting is allowed
0.05            .999
0.3             .980
0.5             .950
0.7             .925
1.0             .9
*END
C
C      define cost functions
```

```

C      three points will be used
*COST
*CFUNC          3
0.1  2.0   50.  1.0
0.5  3.11  50.  1.0
1.0  4.0   50.  1.0
*END
C
C      define the output points
C      (areas for which risk is to be computed)
C      ten points are used
*XCOUT          10
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
*END
C      end risk file
*END

```

4.2 Performance

A second analysis method in RISK is based on performance. In this approach, the risk with respect to performance is adjusted by changing the primitive random variables to meet a user-specified probability of failure at user-specified design values x . This is, in essence, part of the design process.

The input requirements are the P_f function, which can be computed automatically by PFEM, the random variables statistics and probabilistic sensitivities, and the design requirements. The P_f function is input with the *XPF keyword as before. The random variable statistics and probabilistic sensitivities are input with the *RVDEFINE keyword. The design requirements are input under the *PERFORMANCE keyword by using the *DREQ and *DVAR keywords. NESSUS/RISK will compute the necessary changes in the random variable means and standard deviations needed to meet the requirements. Each random variable will be adjusted independent of the others, and the mean and standard deviation are adjusted independently.

The algorithms used in RISK are described in Appendix C (by T. Tornø and H. Millwater) of the October 1990 PSAM monthly report.

Assumptions:

The user should be aware of the assumptions involved in the implemented algorithms. First, for nonlinear functions and non-normal distributions, the computed design changes are only a first order estimate. An iterative process is needed for an accurate solution as outlined in the technical write-up. Secondly, the probabilistic sensitivities are assumed constant, whereas, in general they will vary over the range on the failure function. It is assumed that when utilizing RISK/Performance the user will be focusing on a specific area of the failure function, e.g. left tail, and the probabilistic sensitivities will be approximately constant. Thirdly, the technical write-up details an algorithm for a variety of non-normal distributions. In this version of NESSUS/RISK, a normal distribution is assumed for all random variables.

Example:

This problem has two random variables, x_1, x_2 with statistics

$\mu_1 = 25. \quad \sigma_1 = 4. \quad \text{normal distribution}$

$\mu_2 = 10. \quad \sigma_2 = 3. \quad \text{normal distribution}$

and probabilistic sensitivity factors

$\alpha_1 = -0.8$

$\alpha_2 = 0.6$

The failure function is defined by two points in the left tail,

$x = 1 \quad P_f = 1.35E-3$

$x = 2 \quad P_f = 0.1$

The design requirements are:

$P_f = 2.33E-4$ at $x = 1$

and

$P_f = .001$ at $x = 1.5$

Both the mean and standard deviation will be changed independently to meet the design requirements if needed.

The input file would look like:

```
*RISK
*PERFORMANCE
    *DREQ          2          two points are requested  1.0  2.33E-4
    2.0  0.001
C
C          both mean and sigma to be varied
    *DVAR          BOTH
*END
C
    *XPF          same are for cost
    *TABLE          2          two points
    1.0          1.35E-3
    2.0  0.1
*END
```

```

C
C      define random variables
C
C      note: presently all ran vars assumed normal
*RVDEFINE
  *DEFINE    1
    25.  4.   NORMAL
    -0.8
  *DEFINE    2
    10.  3.   WEIBULL
    0.6
*END
C
C      end risk input
*END

```

4.3 USER-Defined Risk Criteria

An option for a user-defined criteria for risk is provided through a user-written subroutine, USRRSK. The probability of failure function and the random variable statistics can be used with this option.

4.4 SSME Risk Research

Rocketdyne performed a risk and cost model survey of SSME hardware. The report is enclosed in Appendix D.

5 SUPPORTING TECHNOLOGY

5.1 FPI Enhancements

To support accurate and fast structural reliability analysis, an improved fast probability analysis method was developed. This improved method combines the fast probability integration concept, the convolution theorem, and the fast Fourier transform technique. The general procedure of this fast convolution method (FCM) consists of the following steps: (1) identify the most probable point of a limit state, (2) establish a quadratic surface around the most probable point, (3) transform the quadratic surface to a linear surface, and (4) apply a fast Fourier transform technique to provide a fast convolution solution. Detailed discussions of this methodology and test examples are contained in Appendix E.

The method can solve previously encountered numerical problems associated with highly non-normal distributions or very large coefficient-of-variation distributions. Also, the method is more accurate than the previous fast probability integration (FPI) method. The following example represents one of the problems for which the normal transformation distorts the original limit state (in the X -space) drastically such that the limit state cannot be well-approximated by a second-degree polynomial in the transformed u -space. As a result, the standard first-order reliability method (FORM) and the second-order reliability method (SORM) may produce significant errors.

Example:

The g -function is:

$$g = R - S$$

where

$$R \sim \text{Lognormal} (\text{mean} = 20; \text{Cov} = 0.25)$$

and S has a bi-modal PDF defined as:

$$f_S(s) = (0.99) \cdot \phi\left(\frac{s - \mu_1}{\sigma_1}\right) + (0.01) \cdot \phi\left(\frac{s - \mu_2}{\sigma_2}\right)$$

where

$$(\mu_1, \sigma_1, \mu_2, \sigma_2) = (10, 2, 40, 2)$$

The reliability analysis results are summarized in the Table 5.1. Figure 5.1 shows the exact limit state and the full second-order approximation in the u -space. The FCM (X -space) method produces near exact solution because the g -function is linear. The results suggest that the g -function in the u -space cannot be adequately approximated by a quadratic function because of the nonlinear normal transformation.

Table 5.1
Reliability Results

Method	FORM	Previous Method ¹	SORM ²	FCM u -space	FCM X -space	Exact ³
Probability of Failure	1.567e-2	Numerical Problem	1.517e-2	1.507e-2	2.347e-2	2.307e-2

¹ Advanced first-order FPI method

² Based on formula developed by Breitung (see Appendix E)

³ Based on importance sampling method

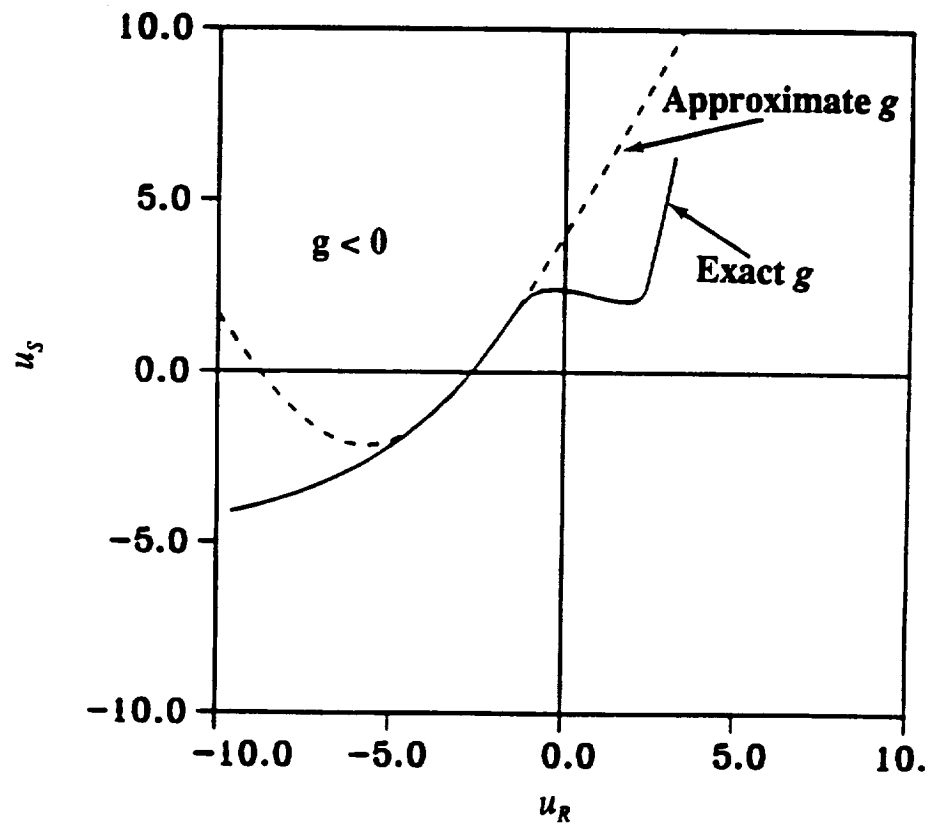


Figure 5.1 Illustration of Exact and Approximate Limit States

5.2 Nessus/Probabilistic Boundary Element Method

A probabilistic boundary element method program, PBEM, has been developed as an alternative to finite elements. PBEM is based on a simple perturbation interface for the BEST3D, version 3, boundary element program. This interface allows the user to define structural random variables, compute sensitivities, and perform advanced mean value updates, i.e., "Moves." The automated coupling with FPI has not been developed, but the concepts are shown here. A simple example problem using PBEM has been successfully completed. The input format for PBEM is covered in detail in Appendix F.

The tasks completed include a complete BEM perturbation analysis capability using version 3.0 of BEST3D. The development of an automatic perturbation database capability will be completed in FY'91. In essence, the tasks completed are as follows:

- a) The new version of BEST3D has been incorporated with an interface for performing the perturbation analysis for all types of the deterministic problems BEST3D can handle. This part of the program is called PBEM.
- b) The automated moving of the design points for advanced mean value method can be performed using the module MOV. This is not completely automated in the sense the FPI data deck of the perturbed solutions has to be separately defined. However, this ability to move the design points automatically using MOV helps in saving considerable time and manual interaction.
- c) The existing program modules are sufficient to perform all types of probability analysis for which a deterministic analysis can be performed using BEST3D. An example involving a cantilever beam has been successfully completed.

The present version of PBEM is compatible with the current version of PFEM. However, a module for automatic FPI database development and tying it with PBEM and MOV is required. It is expected that these tasks will be completed in the next fiscal year.

As described below, a cantilever beam under tip load has been solved successfully, using the new version of PBEM. Note that the present version of PBEM can also be used to perform probabilistic analysis of nonlinear, dynamic, and heat transfer problems.

VALIDATION PROBLEM

Cantilever Beam Under a Tip Load

The problem under consideration is static, and the PBEM is used to get the perturbed solution. The model uses 28, 8-noded isoparametric quadrilateral boundary elements. The dimension of the beam and the loading conditions are as shown in the Figure 5.2, and the details of the random variables are given in Table 5.2. The tip displacement of the cantilever beam is the response variable. The cumulative distribution function of the tip displacement is shown in Figure 5.3 and compares well with the exact solution from Monte Carlo. The CDF shown in Figure 5.3 is only for the static analysis; however, all other problems involving dynamic, thermal, and inelasticity could similarly be determined.

Table 5.2
Total Number of Random Variables = 4

Variables	Distribution	Mean	Standard Deviation
Distributed Load	Normal	100.0 psi	10.0 psi
Young's Modulus	Lognormal	3.0E + 7 psi	3.0E + 6 psi
Length of the Beam	Lognormal	16.0 in.	0.8 in.
Depth of the Beam	Lognormal	4.0 in.	0.4 in.

Tip displacement =

$$\delta = \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI}$$

or

$$\delta = \frac{2Px^3}{Etd^3} - \frac{6Pl^2}{Etd^3} + \frac{4Pl^3}{Etd^3}$$

where

- w = Uniformly distributed load
- E = Young's modulus
- v = Poisson's ratio
- t = Thickness of the beam
- d = Depth of the beam
- l = Length of the beam

For the present data, the tip displacement for the beam shown in Figure 5.1 is = 0.003143 inches, which is 0.98 x (displacement from BEM). As a result, the beam solutions are calibrated using a factor of 0.98.

Table 5.3
Tip Displacement (Z direction, in inches)
Due to Deterministic Solution and Perturbation Solution

Deterministic solution	0.34829e-02
Perturbed solution: "()" is number of standard deviations	
Load perturbation.(1)	0.35178e-02
Material (E) perturbation.(1)	0.24484e-02
Depth perturbation.(0.1)	0.34763e-02
Length perturbation.(0.1)	0.34879e-02

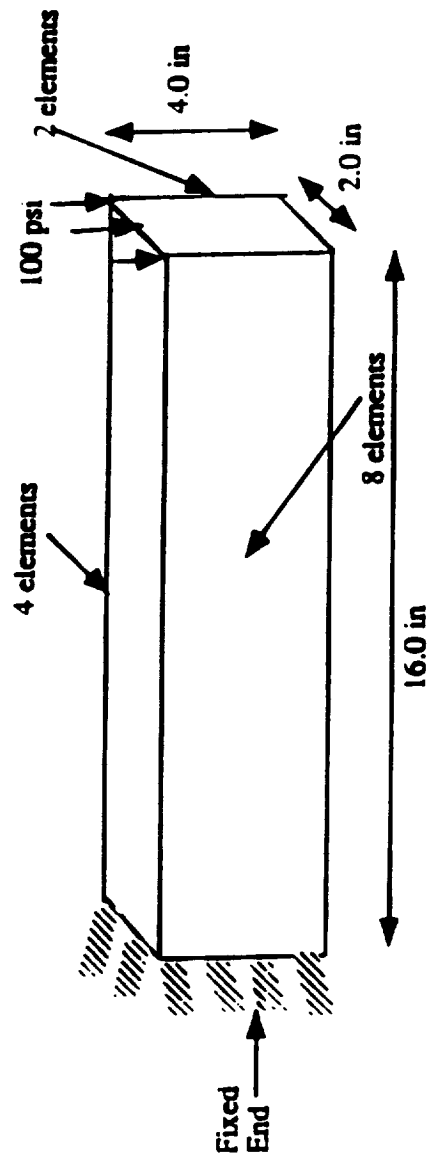


Figure 5.2 Cantilever Beam Under a Tip Load

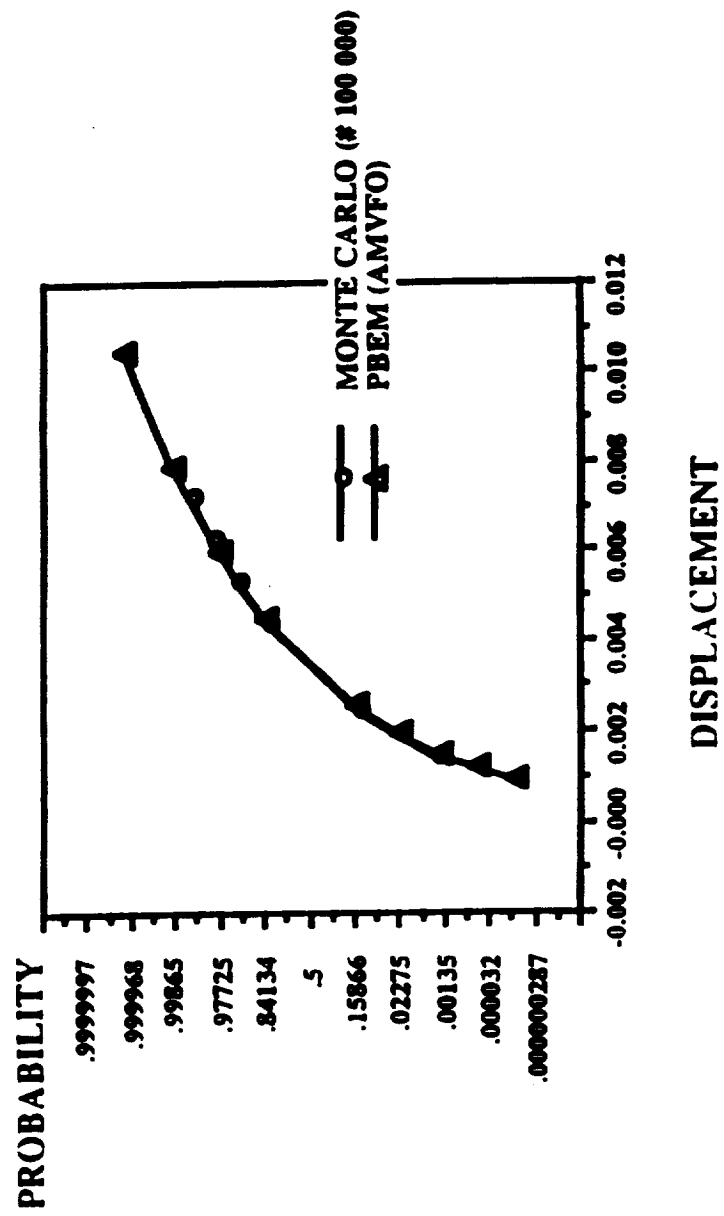


Figure 5.3 Cumulative Distribution Function of Tip Displacement

5.3 Expert System

The Expert system was updated to include help screens for probabilistic options such as perturbation size, convergence criteria, analysis type, analysis method, g function, and confidence intervals. The help screen for perturbation sizing and convergence criteria is shown below. The help screens for other options are contained in Appendix G.

Vanderbilt University has been working on defining a new interface to NESSUS called NESSUS/SHELL. SHELL is a graphics-oriented, menu-driven system that will assist the user in defining a probabilistic input given deterministic input. A preliminary write-up for SHELL is given in Appendix H.

HELP SCREEN FOR PERTURBATION SIZING

*PERT

This keyword is used to determine the amount by which random variable(s) will be perturbed. The purpose of this procedure is to determine the response sensitivities. NESSUS uses an iterative perturbation algorithm to compute the response sensitivities; therefore, this option is closely tied with the convergence criteria on the *ITER card.

Several considerations should be kept in mind when using this option:

- i) a "small" perturbation will give a better estimate of the local sensitivities.
- ii) the perturbation should not be so small that numerical noise washes out the effect of the perturbation.
- iii) smaller perturbations will converge quicker.
- iv) too large a perturbation may result in nonconvergence of the algorithm.

Perturbation Sizing

As a starting point, we recommend a perturbation size of $0.1 \times$ the random variable standard deviation with a relative error convergence criteria on the *ITER card of 0.005. This choice depends on the random variable standard deviation, and for problems with high coefficient of variation, the user may want to choose a perturbation of approximately .1 percent of the random variable mean.

The appropriate perturbation size is of course problem dependent but some generalizations can be made. Note, however, experience is the best guide!!!

Perturbations containing loads make no difference as loads affect only the right-hand side and convergence will be achieved in 1 iteration.

The structure is oftentimes fairly sensitive to geometric parameters such as length, thickness, etc. Thus, perturbations with respect to geometric random variables may need to be relatively small.

In general, if the user is aware that the structural response is sensitive to the random variable to be perturbed then the perturbation used may need to be relatively small.

The user should always check the sensitivities which are computed to see if they make sense, i.e., is the sign correct? Is the magnitude in comparison with other random variables sensitivities what one would expect?

Theoretically, the size of the perturbation should not affect the CDF results. The perturbations should be small enough to pick up accurately the local sensitivities, but not too small such that the sensitivities get lost in numerical noise.

Guideline of Random variable sensitivities

RHS affects only. Perturbation sizes and convergence criteria have no effect.

Force

Pressure

Distributed load

Initial conditions. May be very sensitive to random variable perturbations.

Acceleration

Displacement

Velocity

Geometric effects. May be very sensitive to random variable perturbations.

Coordinates

Beamsections

Material properties. Most likely average sensitivity. Often, these random variables are often the more significant random variables.

Properties

Orientation

Damping - may have high sensitivity

Yield Function

Temperature - effects temperature dependent materials

Stiffness affects. Most likely average sensitivity.

Springs

Random vibration/Harmonic Excitation. Some affect RHS only, others will affect the stiffness matrix.

PSD - Changes in PSD magnitude and shape will affect the RHS only; therefore, the perturbation sizes and convergence criteria will have no effect.

Harmonic - changes in amplitude should affect RHS only. Changes in frequency

will have a nonlinear effect with most likely average sensitivity.

***ITER**

This option is used for defining convergence criteria with respect to the mixed-iterative formulation, material or geometric nonlinearities and the perturbation algorithm.

With regards to the perturbation algorithm, the maximum allowable relative error in the residual is most often used. The other criteria are often left blank. As a starting point, the maximum allowable relative error is recommended to be 0.001 to 0.005 . Good results have been achieved with 4 to 5 iterations in the perturbation algorithm. If convergence is not achieved within 4 to 5 iterations the user should probably reduce the perturbation size. A good upper bound for the maximum number of iterations is 10.

The user should always check the sensitivity results carefully!

APPENDIX A

NESSUS 5.0 Release Notes

Southwest Research Institute

**NESSUS 5.0
RELEASE NOTES**

Prepared By

Southwest Research Institute

January 31, 1991

SwRI Project 06-3285

NAS3-24389

**Probabilistic Structural Analysis Methods
for Select Space Propulsion System Components-II**

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1.0 Introduction

This document contains instructions on using version 5.0 of NESSUS and supercedes previous NESSUS manuals.

A significant number of new features have been added in version 5.0 of NESSUS. Each of the items listed below is discussed further in the following sections.

- Automated solution procedures
 - i) The Advanced Mean Value iteration (AMV+) algorithm (PLEVELS) is implemented in its entirety and automated within NESSUS.
 - ii) An AMV based reliability algorithm (ZLEVELS) is implemented in its entirety and automated within NESSUS.
- Improved input format
- Combined stress and resistance models
- Post-processing of FEM results such as load combination rules, principal stresses, coordinate transformations, maximums and minimums available with NESSUS.
- Coefficient random variables implemented into NESSUS/FEM.
- Risk module which computes risk with respect to cost, performance and a user-defined criteria.
- User-defined hooks to allow the user to define a new material resistance model, post-process FEM results, etc.

Automated solution procedures

Two different AMV+ algorithms have been developed and implemented in NESSUS. One algorithm is used when the probability is prescribed and the corresponding response value z is to be computed. This algorithm is termed the AMV+ p -level procedure. NESSUS 5.0 performs automated iterations until a user-specified tolerance is reached.

The other algorithm, which is new to version 5.0, is used when the z value is prescribed and the corresponding probability is to be computed, and is termed the AMV+ z -level procedure. Again, iteration is automatic until a user-specified tolerance is reached. A summary of both algorithms is given below.

AMV+ p -level Procedure

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The AMV+ p -level procedure is used to compute the response corresponding to a specified probability, or p -level. In NESSUS, the method is used when points along the entire range of the CDF are to be computed or the response at specified probability levels is to be computed. This procedure is selected by setting *ANALTYPE = 0 or 2 in the FPI input section of PFEM.

Table 1 gives a schematic of the method, along with the number of finite element solutions that are required to perform a first order probabilistic analysis. In Table 1, N is the number of input random variables and M is the number of p -levels. MVFO refers to "mean value first order," and AMVFO refers to "advanced" MVFO. In an MVFO analysis, $(N+1)$ sensitivities are computed and used to construct a linear performance function (g -function) about the mean values of the random variables. In step 2, an update (or "move") is performed which entails a finite element solution to update the response value at each of the M probability levels.

Table 1. Schematic of AMV+ p -levels Procedure

Iteration	Step	Analysis	Number of Finite Element Solutions
0	1	MVFO	$N + 1$
0	2	AMVFO (Move)	$+ M$
1	1	First Order	$+ (N + 1) \times M$
1	2	Move	$+ M$
2	1	First Order	$+ (N + 1) \times M$
2	2	Move	$+ M$
.	.	.	.
.	.	.	.
.	.	.	.

A fully automated AMV+ p -level algorithm has been implemented in NESSUS. The user inputs an allowable number of iteration and a convergence tolerance. At each p -level, NESSUS continues iteration until either *i*) the allowable number of iterations is reached, or *ii*) the relative change in z is within the convergence tolerance.

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AMV+ z-level Procedure

The AMV+ z-level procedure is used to compute the probability corresponding to a specified response or z-level. In NESSUS, the method is primarily used for reliability calculations, where p must be calculated for a specific value of z .

The key to efficient reliability analysis is the ability to compute quickly the location of the most probable point (MPP). To reduce the number of finite element solutions required while searching for the MPP, an algorithm has been devised and implemented to estimate a good starting MPP. Once the estimate is obtained, a straightforward iteration procedure is used to converge on the desired MPP. It should be noted that the sole purpose of the algorithm used to obtain the starting MPP is to minimize the number of finite element solutions, and can be modified or adapted as experience warrants.

The AMV+ z-level procedure works in "u-space," which is related to p by $2p = 1 + \text{erf}(u/\sqrt{2})$, and "erf" is the error function. An outline of the AMV+ z-level procedure is given below:

A. Compute an initial estimate for the most probable point.

1. Estimate the probability level u using $u = az^2 + bx + c$, where a , b , and c are determined from:
 - i) u at the mean value of z ,
 - ii) $\partial u / \partial z$ at the mean value of z , and
 - iii) the AMV result (move) at $u = +5$ or -5 , depending on whether the z-level is greater than or less than its median, respectively.
2. From the probability level u computed in Step A-1, compute the MPP using FPI and the response using a move (AMV procedure).
3. Compute a new estimate of the probability level u using $u = az^2 + bx + c$, where a , b , and c are determined from:
 - i) u at the mean value of z ,
 - ii) u at the MPP obtained in Step A-2, and
 - iii) the AMV result (move) at $u = +5$ or -5 , depending on whether the z-level is greater than or less than its median, respectively.
4. From the probability level u computed in Step A-3, compute the starting MPP using FPI.

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B. Given a starting u and MPP:

- i) obtain sensitivities at the MPP, and
- ii) estimate an improved u and MPP for the input z -level using FPI.

C. Repeat step B until either:

- i) the allowable number of iterations is reached or
- ii) the relative change in u is less than the convergence tolerance.

Improved Input Format

The input format for PFEM of NESSUS 5.0 is similar to that of version 4.2, but is expanded and more user-friendly. In particular, the input is now column and case independent. A number of new key words have been added reflecting new code capabilities.

Combined Stress and Resistance Models

A significant new feature of NESSUS 5.0 is the ability to compute the probabilistic response of general functions that are a combination of finite element quantities and material resistance. This is done by computing sensitivities with respect to load, material properties, etc., of the *combined* response.

A very flexible framework is provided for user-defined response functions. Some examples which can be easily programmed include: Campbell diagram, high/low cycle fatigue models, and fracture models.

Post-processing of FEM Results

In many cases, the user wishes to manipulate the FEM results prior to probabilistic analysis. Examples may be to implement load combination rules, principal stresses, coordinate transformations, maximums and minimums. This can now be done easily in NESSUS 5.0. A library of pre-programmed functions is included in NESSUS 5.0. In addition, the user can program his own transformations through user-defined subroutines.

Coefficient Random Variables Developed within NESSUS/FEM

A new type of random variable labeled "COEF" for coefficient has been added to NESSUS/FEM. This random variable will be perturbed the same as other random variables in FEM; however, it has no direct effect on the structure, but does get passed into the user-defined material routines. Thus, constants such as the material power coefficient can be considered random

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and passed into the material routines. The COEF random variables will provide a mechanism whereby the material can be degraded as a result of the structural response. COEF should be useful for creep, random stress-strain curves, fatigue, and any material models.

NESSUS/RISK

A risk module has been added to NESSUS, called NESSUS/RISK, which will compute the risk with respect to cost, performance, and a user-defined criteria. A summary of the algorithms is given below. More information, including example problems and input keywords, is given in section 5.

COST

Risk with respect to cost is computed using the formula

$$\text{Risk/Cost} = C_0(x) + P_f(x) * C(x) + (1-P_f) * NC(x)$$

where x is the allowable response in a reliability analysis such as stress, displacement, cycles to failure; C_0 is the initial cost as a function of x ; P_f is the probability of failure as a function of x ; and $NC(x)$ is the probability of nonfailure as a function of x . The $P_f(x)$ curve may be the CDF computed by NESSUS; however, in general, x can be any design parameter including non-random variables.

PERFORMANCE

A second analysis method in RISK is based on performance. In this approach, the risk with respect to performance is adjusted by changing the primitive random variables to meet a user-specified probability of failure at user specified design values x . This is in essence part of the design process.

The input requirements are: the probability of failure function, P_f , the random variable statistics and probabilistic sensitivities and the design requirements. NESSUS/RISK will compute the necessary changes in the random variable means and standard deviations needed to meet the requirements. Each random variable will be adjusted independent of the others, and the mean and standard deviation are adjusted independently.

USER

An option for a user-defined criteria for risk is provided through a user-written subroutine, USRRSK. The probability of failure function and the random variable statistics can be used with this option.

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RISK can be run stand-alone or in conjunction with PFEM. Analysis with PFEM is fully automated from a single input file.

New Files

The user will notice three new files in NESSUS 5.0. `jobname.PDB2` contains the finite element sensitivities at most probable points, whereas `jobname.PDB` contains sensitivities around the mean. `jobname.PDB2` is a scratch file used for the automated AMV+ algorithm. The user should have no need for this file.

The second file is the probabilistic storage file `jobname.PSF`. This file will store the probabilistic results; however, it is not operational for NESSUS 5.0.

The third file is the risk input file `jobname.rsk`, and is created by NESSUS when performing a risk computation within a PFEM analysis.

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2.0 Summary of Changes from Version 4.2

This section describes overall changes to the NESSUS code from the previous 4.2 release. The new features and enhancements present in version 5.0 have been implemented in the NESSUS/PFEM module and the newly created RISK module.

Stand-alone operation of both the NESSUS/FEM and NESSUS/FPI modules is identical to that in previous versions with the following exceptions:

1. The NESSUS/FPI input is now column independent.
2. A new random variable type (COEF) has been added to NESSUS/FEM.

NESSUS/FEM Users Manual update pages are included in Appendix A for the new COEF random variables. The purpose of these new random variables will be discussed later in this document.

Specific new capabilities incorporated in NESSUS 5.0 include:

1. a fully automated iteration algorithm using the AMV+ procedure to obtain either (i) the response at a given probability or (ii) the probability at a given response, both to within a user-specified tolerance;
2. a very general capability for analyzing complex performance functions by combining both load and material/resistance variables; and
3. a capability whereby the material properties may be degraded as a function of stochastic load and material/resistance variables during the course of the finite element calculations, and
4. a risk module that will compute the risk with respect to cost and performance. Risk can be run stand-alone or with a PFEM analysis.

The structure of the new NESSUS/PFEM input deck is shown below. The input is divided into four major sections:

1. probabilistic analysis input,
2. finite element analysis input,
3. fast probability integration input, and
4. risk analysis input.

Sections 1-3 are required input for a PFEM analysis. Section 4 is optional and is only required when running a risk analysis with PFEM.

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```
*PFEM
    Probabilistic Analysis Input
*END
*FEM
    FE Analysis Input
*END
*FPI
    FPI Analysis Input
*END
*RISK
    Risk Analysis Input (optional)
*END
```

NESSUS/PFEM Input Structure

Note that each major input section ends with a ***END** keyword card.

When coupled via a PFEM analysis, the input decks for NESSUS/FEM and NESSUS/FPI codes are only slightly changed from their stand-alone versions; random variable definitions (***DEFINE** and ***PERT** cards in FEM and ***DEFRAVR** in FPI) are now input in the probabilistic analysis input section, ***PFEM**. Also, NESSUS/FPI input is now read in column-independent format. These changes will reduce errors and the possible ambiguity that can result when multiple definitions of the same random variable are given in the same input deck. The random variable input cards also now accept an 8-character arbitrary name for each random variable.

To describe the approach used in NESSUS 5.0 to allow general reliability analysis, it is useful to review the methodology used in the earlier versions of NESSUS. Previously, a structural response function $Z = Z(X)$ was defined either by a numerical method such as NESSUS/FEM or by a closed form expression. In a probabilistic structural response analysis, the basic capability of version 4.2, the performance function was formulated as $g(X) = (Z(X) \leq Z_0)$, where Z_0 can be thought of as a limiting value of Z . From this, it can be seen that the performance function is $g(X) = Z(X) - Z_0 = 0$. Point probability estimates were made using specific limits for Z_0 , and the cumulative probability distribution function (CDF) was obtained by simply varying Z_0 and repeating the analysis.

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To define reliability (as opposed to structural response) requires a more general definition of the performance function. The approach used to accomplish this in NESSUS is straightforward. Instead of interpreting $Z = Z(X)$ as being solely the structural response, its definition is extended to include the effects of both the response and material/resistance model. Specifically, $Z(X)$ is formulated to represent the *overall* response function. For a structural reliability analysis, $Z(X)$ will be a function to predict a particular failure mode. (Note that the probabilistic structural response analysis version 4.2, is a subset of the capabilities in NESSUS 5.0.)

A very general framework has been implemented in NESSUS 5.0 for defining performance functions. Work is currently underway to code in a library of response functions, and user programmable subroutines have been provided to allow "home grown" performance functions to be defined.

In summary, the performance function in version 4.2 could be either a structural response (finite element) or a closed-form response (PAAM). In version 5.0, the response can be a combination of structural response and closed form. Thus, material resistance models can be coupled with the structural response for reliability calculations.

A risk module, as mentioned in the introduction, has been implemented into NESSUS 5.0. The risk module can be run within a PFEM analysis. The input for the RISK module, see section 5, is the same as when RISK is run stand-alone, except the probability of failure function is supplied by PFEM. The probability of failure function is essentially the CDF computed by PFEM.

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3.0 NESSUS/PFEM Input

The probabilistic analysis section (*PFEM) is divided into four major input sections:

1. z-function definition (*ZFDEFINE) input,
2. random variable definition (*RVDEFINE) input,
3. mean value analysis definition (*MVDEFINE) input, and
4. advanced mean value analysis definition (*AMVDEFINE) input.

The order of input of each section is arbitrary. To summarize, the *ZFDEFINE input block contains the input necessary for defining the response function and for including resistance variables in the probabilistic analysis. The *RVDEFINE input block contains the random variable definitions. The *MVDEFINE and *AMVDEFINE input blocks define the mean value and advanced mean value analyses, respectively.

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Figure 1. PFEM Input Deck Structure.

```

*PFEM
*ZDEFINE
  *COMPUTATIONALMETHOD lcnod nvars
  jvar(1),i=1,nvars
  *EXPLICITVARIABLES nvars
  jvar(1),i=1,nvars
  *ZFUNCTION izfunc nzcoef
  zcoef(1),i=1,nzcoef
  *VARIABLE lcvr
  PESPTYPE jresp
  LAYER layer
  *CONDITIONLIST ncond
  jcond(1),i=1,ncond
  *MODEL nmod
  jmod(1),i=1,nmod
  *COMPONENTLIST ncomp
  jcomp(1),i=1,ncomp
  *TRANSFORMATION jtran
  jaxis(1) ang(1)
  jaxis(2) ang(2)
  jaxis(3) ang(3)
  *FEFATION lopt noccoef
  locoef(1),i=1,noccoef
  *OPERATION
  ENO
  *ZFUNCTION
*END
*HVDDEFINE
  *DEFINE lrvnum
  rname
  mean sdev dist
  distribution_data
  type
  datablock
  *FEF0FB jpart
  jpar(1) shift(1)
  jpar(2) shift(2)
  .
  .
  .
  jpar(n) shift(n)

```

```

*END
*HVDDEFINE
  *DATATYPE
  *RESPTYPE
  *CONDITION (Replaces *INCR,*MODE,*HARM)
  *NODE
  *COMPONENT
  *LAYER
  *PERTURBATIONS
  *RANVARIABLES
  *RESTART
  *END
  *PAM
*END
*HVDDEFINE
  *CONDITION (Replaces *INCR,*MODE,*HARM)
  *NODE
  *COMPONENT
  *LAYER
  *MOVE
  *END
  *PAM
  *ITERATION (New)
*END
*FEM
  *NESSUS/FEM input
*END
  *FPI
  *NESSUS/FPI input
*END

```

3.1 Input Features

- Input is keyword driven for readability and can appear anywhere on the input line, allowing for indentation. All alphanumeric input is matched on the first four characters. Comment lines are designated by inputting a "C" in column 1.
- Each input line is first read as a character string, from which the input information is then parsed. The result is that the user is given warning and error messages if an input line is in error, rather than crash the program. The exception is when a series of real or integer data is to be input, in which case a standard free format read is used; this allows the user to input data across multiple lines if desired.
- Random variables are input in the PFEM input block as opposed to being defined in both the NESSUS/FEM and NESSUS/FPI input blocks. Also, a distribution and a descriptive name is input for each random variable.

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3.2 Z-Function Definition (*ZFDEFINE)

In NESSUS 5.0, random variables are not defined in terms of whether they are "stress" or "resistance" variables. Instead, they are defined as either "computational" or "explicit" random variables. Computational random variables are those which affect the response calculation directly, i.e., the finite element analysis. Explicit random variables are those which are non-computational, i.e., not FEM random variables. (Random variables may also be both computational *and* explicit.) In version 5.0, NESSUS/FEM will be the computational method used. Future versions will have an option for the boundary element method to be the computational tool.

From the computational and explicit random variables, a very general response function, Z, can be defined. This Z-Function is defined in this section.

Summary of *ZFDEFINE keywords

- *ZFDEFINE Signals beginning of z-function definition.
- *ZFUNCTION Selects the response function.
- *UZFUNCTION Signals that the user-defined z-function is to be used.
- *COMPUTATIONALMETHOD Selects the computational method and random variables.
- *EXPLICITVARIABLES Selects the explicit random variables.
- *CVariable Defines a computational response variable and operations.
- *END Signals end of z-function definition.

*ZFDEFINE

*ZFDEFINE signals the start of the z-function definition. Following keywords will be interpreted as z-function keywords until the *END keyword is reached.

*ZFUNCTION izfunc nzcoef
zcoef(i), i=1, nzcoef

*ZFUNCTION is used to select the response function. If this card is omitted, the structural response value selected in the *MVDEFINE input section will be used. (This keyword is considered optional.)

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Response function models are selected (1) from a library of pre-programmed functions available in NESSUS, or (2) from a user-programmed subroutine. If the `UZFUNCTION` keyword card is omitted, then `izfunc` will refer to one of the pre-defined response functions available in NESSUS. If the `UZFUNCTION` keyword card is present, then the user subroutine `UZFUNC` is called, and the user is required to provide coding to define the response function. The `izfunc` parameter is passed into the `UZFUNC` subroutine, so that the user can develop a personal library of response functions.

`izfunc` is the response function number.

`nzcoef` is the number of coefficients to be made available to the response function definition.

`zcoef` is the list of real coefficients to be made available to the response function. (Required if `nzcoef` is greater than zero.)

*UZFUNCTION

`UZFUNCTION` signals that the user-subroutine `UZFUNC` (User ZFUNCTION) is to be used to define the response model. The default is to not call `UZFUNC`.

NESSUS contains a library of pre-defined response models for often used functions, such as Paris crack growth, Low Cycle Fatigue, etc. If the `UZFUNCTION` keyword card is omitted, then `izfunc` will refer to one of the pre-defined response models available in NESSUS. If the `UZFUNCTION` keyword card is present, then the user subroutine `UZFUNC` is called, and the user required to provide coding to define the response model. The `izfunc` parameter is passed into the `UZFUNC` subroutine so that the user can develop a personal library of defined response models.

*COMPUTATIONALMETHOD `icmod nvars`
`jivar(i), i=1, nvars`

The *COMPUTATIONALMETHOD keyword is used to select the computational method and the associated random variables. (Optional-use only if a finite element model is included.)

`icmod` is the computational method. (Required if `nvars` is greater than zero)

`icmod = 1 - NESSUS/FEM`

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nsvars is the number of computational random variables. (Required if keyword is present.)

jsvar is an integer list of computational model random variable numbers. (Required if **nsvars** is greater than zero.)

***EXPLICITVARIABLES nsvars**
jsvar(i), i=1, nsvars

***EXPLICITVARIABLES** is used to identify the explicitly defined random variables (Optional).

nsvars is the number of explicitly defined random variables. (Required if keyword is present.)

jsvar is an integer list of explicitly defined random variable numbers. (Required if **nsvars** is greater than zero.)

***CVariable icvar**

***CVariable** (Computational VARIABLE) is used to define a new response variable that is a function of the computational response variables obtained from the computational (e.g., FEM) analysis. The definition for each response variable is given on a series of "unstarred" keywords within the ***CVariable** input block and must end with an **END** statement.

Some instances when a ***CVariable** would be used effectively in specifying a new response variable include defining a (1) displacement measure in a different coordinate system, (2) maximum principal stress occurring within a group of nodes in the finite element mesh, or (3) root-mean-square response computed from a selected set of spectral case results.

Defaults are defined for each **CVariable** parameter and are indicated in the descriptions below.

icvar is the computational random variable number

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Summary of *CVARIABLE keywords

Signals beginning of the *CVARIABLE definition.

*CVARIABLE	Selects a response type.
RESPTYPE	Selects a layer.
LAYER	Specifies a list of conditions.
CONDITIONLIST	Specifies a list of nodes.
NODELIST	Specifies a list of components.
COMPONENTLIST	Selects a NESSUS Pre-defined operation.
OPERATION	Selects a user-defined operation.
UOPERATION	Defines a coordinate transformation.
TRANSFORMATION	Signals the end of the *CVARIABLE definition.
END	

A description of each *CVARIABLE keyword is given below:

RESPTYPE jresp

RESPTYPE is used to select the response type. Valid response types are listed below. If RESPTYPE is omitted, jresp will be set to the response type selected in the MVDEFINE section.

jresp is the response key. Valid keys are listed below. A detailed explanation is provided in Section 4.

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Valid Response Keys

1 = TOTAL DISPLACEMENT	36 = THE FREQUENCY IN CYCLES PER TIME
2 = TOTAL STRAIN	51 = REAL DISPLACEMENT
3 = TOTAL STRESS	52 = REAL STRAIN
11 = PLASTIC STRAIN	53 = REAL STRESS
12 = BACKSTRESS	61 = IMAGINARY DISPLACEMENT
13 = CREEP STRAIN	62 = IMAGINARY STRAIN
14 = THERMAL STRAIN	63 = IMAGINARY STRESS
17 = GENERALIZED STRAIN	71 = THE AMPLITUDE OF THE DISPLACEMENT
18 = GENERALIZED STRESS	72 = THE AMPLITUDE OF THE STRAIN
21 = MATERIAL STATE VARIABLES	73 = THE AMPLITUDE OF THE STRESS
25 = VELOCITIES	81 = THE PHASE OF THE DISPLACEMENT
26 = ACCELERATIONS	82 = THE PHASE OF THE STRAIN
30 = THE EIGENVALUE FOR THE MODE	83 = THE PHASE OF THE STRESS
31 = MODAL DISPLACEMENT (EIGENVECTOR)	91 = MEAN SQUARE DISPLACEMENT
32 = MODAL STRAIN	92 = MEAN SQUARE STRAIN
33 = MODAL STRESS	93 = MEAN SQUARE STRESS
35 = THE FREQUENCY IN RADIAN PER TIME	96 = STRESS VELOCITY

LAYER layer

LAYER is used to specify the layer number at which results will be extracted.

In NESSUS 5.0, this keyword is inactive.

layer is the layer number. (Default, layer = 1)

CONDITIONLIST ncond jcond(i), i=1, ncond

CONDITIONLIST defines a list of conditions (increments, modes, or spectral cases) from which to extract results from. If this keyword is included, the OPERATION keyword is required to select an operation to be applied to the extracted data. (An operation is needed to compute

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and return a single implicit response variable from the list of results.) Additionally, the loop over the number of conditions defined on the *CONDITION keyword card in the *MVDEFINE section will be disabled. This is an optional keyword.

An example use of the CONDITIONLIST keyword might be to define a CVARIABLE response variable to be the root-sum-square of a list of modal frequencies.

ncond is the number of conditions.

jcond is the list of condition numbers.

```
MODELIST nnod  
jnode (i) , i=1, nnod
```

MODELIST defines a list of nodes from which to extract results. If this keyword is included, the OPERATION keyword is required to select an operation to be applied to the extracted data. (An operation is needed to compute and return a single implicit response variable from the list of results.) Additionally, the loop over the number of nodes defined on the *NODE keyword card in the *MVDEFINE section will be disabled. This is an optional keyword.

nnod is the number of nodes.

jnode is the list of node numbers.

```
COMPONENTLIST ncomp  
jcomp (i) , i=1, ncomp
```

COMPONENTLIST defines a list of components from which to extract results. If this keyword is included, the OPERATION keyword is required to select an operation to be applied to the extracted data. (An operation is needed to compute and return a single implicit response variable from the list of results.) Additionally, the loop over the number of components defined on the *COMPONENT keyword card in the *MVDEFINE section will be disabled. This is an optional keyword.

ncomp is the number of components.

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jcomp is the list of component numbers.

```
OPERATION iopt nocoef  
ocoef(i), i=1, nocoef
```

OPERATION is used to select an operation to be applied to the extracted data. If iopt is 0 or if this card is omitted, the requested response value, specified on the RESPTYPE keyword, is simply returned. If the CONDITIONLIST, NODELIST and/or COMPONENTLIST keywords are used, the user is required to select an operation. (An operation is needed to compute and return a single implicit response variable from the list of results.) The default is no operation.

NESSUS contains a library of pre-defined operations for often used functions, listed below. If the UOPERATION keyword card is omitted, then iopt will refer to one of the pre-defined operations available in NESSUS. If the UOPERATION keyword card is used, then the user subroutine UOPERA is called, and the user is required to provide coding to define the operation. The iopt parameter is available in the UOPERA subroutine so that the user can develop a personal library of defined operations.

iopt is the operation number. If the UOPERATION keyword is not used, iopt is used to select a pre-defined operation (see table below). If the UOPERATION keyword is used, the iopt user-defined operation coded in the UOPERA subroutine is selected.

nocoef is the number of coefficients to be made available to the operation.

ocoef is the list of real coefficients to be made available to the operation. (Required if nocoef is greater than zero.)

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NESSUS Pre-defined Operations

iopt	Operation
1	Algebraic Maximum Value
2	Algebraic Minimum Value
3	Absolute Maximum Value
4	Absolute Minimum Value
5	Square Root of the Sum of the Squares

UOPERATION

UOPERATION signals that the user-subroutine UOPERA (User OPERAtion) is to be used to define the operation. The default is to not call UOPERA.

NESSUS contains a library of pre-defined operations for often used functions such as maximum value, minimum value, etc (see *OPERATION). If the UOPERATION keyword card is omitted, then iopt will refer to one of the pre-defined operations available in NESSUS. If the UOPERATION keyword card is used, then the user subroutine UOPERA is called, and the user is required to provide coding to define the operation. The iopt parameter is available in the UOPERA subroutine so that the user can develop a personal library of defined operations.

TRANSFORMATION jtran
jaxis (1) ang (1)
jaxis (2) ang (2)
jaxis (3) ang (3)

TRANSFORMATION specifies a coordinate transformation that is to be applied to the data. The transformation is applied to the data before being operated on (see OPERATION keyword). In general, transformations are element type dependent; therefore, a complete description of valid transformations is included in Section 4.0. This is an optional keyword.

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`jtran` is the transformation option. Valid options include:

`jtran = 0` - no transformation is applied and the following three lines are not read.

`jtran = 1` - a rotational transformation is performed and the following three lines are read.

`jtran = 2` - principal values are computed and the following three lines are not read.

`{ jaxis(i) ang(i), i= 1, 3 }` define the axis and angle of rotation for the transformation. Rotations are applied in the same order as the `{ jaxis ang }` pairs are input. Angles must be specified in degrees. Valid transformation are given in Section 4.

The default is no transformation.

END

END signals the end of input for the current ***CARIABLE**.

***END**

***END** signals the end of input for the ***ZFDEFINE** section.

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3.3 Random Variable Definition (*RVDEFINE)

***RVDEFINE**

***RVDEFINE** signals the start of the random variable definition. The following keywords will be interpreted as random variable definition keywords until the ***END** keyword is reached.

```
*DEFINE irvnum  
rvname  
mean sdev dist  
distribution_data  
type  
datablock
```

The ***DEFINE** keyword is similar to the ***DEFINE** keyword used in NESSUS/FEM with the added enhancement of defining a distribution type and a descriptive name. (Note, however, that for a PFEM analysis, ***DEFINE** keywords are not input in the NESSUS/FEM input deck.)

irvnum is the random variable number.

rvname is a user-defined, 8-character name for the random variable.

dist is an alphanumeric distribution name (only the first four characters are significant).
Valid names are:

```
WEIBULL  
NORMAL  
EXTREMEVALUE  
LOGNORMAL  
CHISQUARE  
MAXENTROPY  
NESSUS  
FRECHET  
TWEIBULL  
TNORMAL
```

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If `dist = MAXENTROPY`, `TWEIBULL`, or `TNORMAL`, then the two distribution coefficients, as defined in the FPI User's Manual, are read off of the next line.

If `dist = MAXENTROPY` or `NESSUS`, then `k` coefficients, as defined in the FPI User's Manual, are read from the following lines.

If the random variable number `irvnum` is in the `jsvars` list, i.e., `irvnum` is a computational random variable, then the `type` and `datablock` are expected; otherwise they are not input. Documentation for `type` and `datablock` are given in the NESSUS/FEM User's Manual. Examples for `type` are: `FORCE`, `PROP`, `COOR`, etc. The data block is the associated data corresponding to the `type`, e.g., the forces, material properties, and coordinates.

```
*PERTURB jpert  
jpvar1 shift1  
jpvar2 shift2  
jpvarn shiftn
```

*PERTURB defines a perturbation along with the random variable numbers and the amount each random variable is to be perturbed.

Input for this keyword is identical to that in the NESSUS/FEM User's Manual.

`jpert` is the perturbation number.

`jpvar(i)` is the random variable number.

`shift(i)` is the number of standard deviations by which the `jpvar(i)` random variable is changed.

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3.4 Mean Value Analysis Definition (*MVDEFINE)

***MVDEFINE**

***MVDEFINE** signals the start of the mean value analysis definition. The following keywords will be interpreted as mean value analysis keywords until the ***END** keyword is reached.

In this section, the keywords to define the range of the PFEM analyses for the mean value first or second order, (MVFO or MVSO) and the type of data to be analyzed, e.g., stress, or frequency, are defined. The ***MVDEFINE** keywords provide the same functionality as the ***MVFO** keywords in version 4.2. Mean value first order or second order analyses can be performed with PFEM for any number of nodes, modes, components, increments, or harmonic cases in a single analysis.

The user specifies what data to process by defining the beginning and ending parameters for the keywords: ***COND**, ***NODE**, ***COMP**. The beginning parameter defaults to 1 and the ending parameter defaults to the beginning parameter. The parameters define the range of nodes, modes, etc., to process.

Summary of *MVDEFINE keywords

*MVDEFINE	Signals the beginning of the mean value analysis definition.
*DATATYPE	Selects the data type.
*RESPONSETYPE	Selects the response type.
*CONDITIONNUMBER	Selects the condition (increment, modal or spectral) numbers.
*NODENUMBER	Selects the node numbers.
*COMPONENTNUMBER	Selects the component numbers.
*LAYERNUMBER	Selects the plate element layer number.
*PERTURBATIONNUMBERS	Defines the perturbation numbers.
*RANVARIABLENUMBERS	Defines the random variables numbers.
*RESTART	Signals restarting of probabilistic analysis.
*PAAM	Signals approximate method turbine blade model will be used.
*END	Signals the end of mean value analysis definition.

The purpose of these keywords is identical to version 4.2.

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***MVDEFINE**

***MVDEFINE** signals the start of the mean value analysis definition. The following keywords will be interpreted as mean value keywords until the ***END** keyword is reached.

***DATATYPE idata**

This keyword is used to specify the type of data on which to perform the probabilistic analysis.

- = 0 Incremental
- = 1 Eigenvalue
- = 2 Harmonic/spectral

No default. Must be specified for MV analysis.

***COND icond1 icond2**

***COND** selects the beginning and ending condition numbers for the mean value analysis. Condition refers to either incremental, modal, or harmonic/spectral analysis. The condition type is selected with the ***DATATYPE** keyword. NESSUS will perform probabilistic analysis from condition number **icond1** to **icond2**.

- icond1** = beginning condition number for MV analyses.
- icond2** = ending condition number for MV analyses.
- icond1** default = 1.
- icond2** default = **icond1**.

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*NODE	inode1	inode2
--------------	---------------	---------------

***NODE** selects the beginning and ending node numbers for the mean value analysis. NESSUS will perform probabilistic analysis from node number icond1 to icond2.

inode1 = beginning node number of MV analyses.

inode2 = ending node number for MV analyses.

inode1 default = 1.

inode2 default = **inode1**.

*COMP	icompl	icomp2
--------------	---------------	---------------

***COMP** selects the beginning and ending component numbers for the mean value analysis. NESSUS will perform probabilistic analysis from component number icomp1 to icomp2. Allowable component numbers are 0 through 6.

icompl = beginning component number for MV analyses.

icomp2 = ending component number for MV analyses.

icompl default = 1.

icomp2 default = **icompl**.

*LAYER	ilay1	ilay2
---------------	--------------	--------------

***LAYER** selects the beginning and ending layer numbers for the mean value analysis of the plate element (#75). NESSUS will perform probabilistic analysis from layer number ilay1 to ilay2. Allowable layer numbers are through 5.

ilay1 = beginning layer number for MV analyses.

ilay2 = ending layer number for MV analyses.

ilay1 default = 1.

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ilay2 default = ilay1.

***PERT npert**
perts (i) , i=1, npert

***PERT** selects the perturbations to be used in the probabilistic analysis and the order in which the random variables will be perturbed.

npert = the number of perturbations.

perts = is a list of perturbation numbers.

No default. Must be specified for MV analysis.

***RANVAR nranvr**
ranvrs (i) , i=1, nranvr

***RANVAR** selects the random variables to be used in the probabilistic analysis and their order. This is the order the random variables will be presented in the user-written subroutines RESPON and UZFUNC.

nranvr = the number of random variables.

ranvrs = is a list of random variable numbers.

No default. Must be specified for MV analysis.

***RESPTYPE irstyp**

irstyp specifies the response variable to extract from the perturbation database. The allowable options are:

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=01 A TOTAL DISPLACEMENT COMPONENT	=52 A REAL COMPONENT OF THE STRAIN
=02 A TOTAL STRAIN COMPONENT	=53 A REAL COMPONENT OF THE STRESS
=03 A TOTAL STRESS COMPONENT	=61 AN IMAGINARY COMPONENT OF THE DISPLACEMENT
=11 A PLASTIC STRAIN COMPONENT	=62 AN IMAGINARY COMPONENT OF THE STRAIN
=12 A BACKSTRESS COMPONENT	=63 AN IMAGINARY COMPONENT OF THE STRESS
=13 A CREEP STRAIN COMPONENT	=71 THE AMPLITUDE OF THE DISPLACEMENT
=14 A THERMAL STRAIN COMPONENT	=72 THE AMPLITUDE OF THE STRAIN
=17 A GENERALIZED STRAIN COMPONENT	=73 THE AMPLITUDE OF THE STRESS
=18 A GENERALIZED STRESS COMPONENT	=81 THE PHASE OF THE DISPLACEMENT
=30 THE EIGENVALUE FOR THE MODE	=82 THE PHASE OF THE STRAIN
=31 A MODAL DISPLACEMENT (EIGENVECTOR) COMPONENT	=83 THE PHASE OF THE STRESS
=32 A MODAL STRAIN COMPONENT, IF AVAILABLE	=91 A MEAN SQUARE VALUE OF THE DISPLACEMENT
=33 A MODAL STRESS COMPONENT, IF AVAILABLE	=92 A MEAN SQUARE VALUE OF THE STRAIN
=35 THE FREQUENCY IN RADIAN PER TIME	=93 A MEAN SQUARE VALUE OF THE STRESS
=36 THE FREQUENCY IN CYCLES PER TIME	=96 A STRESS VELOCITY VALUE
=51 A REAL COMPONENT OF THE DISPLACEMENT	

No default. Must be specified for MV analysis.

***RESTART** *irest*

This keyword is used to bypass the mean value finite element perturbations. All probabilistic algorithms will be initiated from this point. The user must have available a jobname.pdb perturbation database.

irest specifies the restart option. *irest* = 1 signifies NESSUS to bypass mean value finite element database generation.

***PAAM**

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Signifies NESSUS that the approximate model for the turbine blade is to be used. See the 5th annual PSAM report or the paper, "Application of the Probabilistic Approximate Analysis Method to a Turbopump Blade Analysis," by B. Thacker, R. McClung, and H. Millwater, presented at the 31st SDM Conference, Long Beach, CA, 1990.

***END**

END signals the end of input for the *MVDEFINE section.

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3.5 Advanced Mean Value Analysis Definition (*AMVDEFINE)

*AMVDEFINE

***AMVDEFINE** signals the start of the advanced mean value analysis definition. The following keywords will be interpreted as advanced mean value analysis keywords until the ***END** keyword is reached.

If the ***AMVDEFINE** card is present, an advanced mean value analysis will be performed.

In this section, the keywords to define the range of the PFEM analyses for the advanced mean value with iterations (AMV+) first or second order (AMVFO or AMVSO) procedure are specified. Advanced mean value first order or second order analyses can be performed with PFEM for any number of nodes, modes, components, increments, or harmonic cases in a single analysis.

A significant enhancement in version 5.0 is the automated iteration capability for the p-level and z-level procedures. The ***ITER** keyword and the FPI ***ANALTYPE** keyword controls these algorithms.

The user specifies what data to process by defining the beginning and ending parameters for the keywords: ***COND**, ***NODE**, ***COMP**. The beginning parameter defaults to 1 and the ending parameter defaults to the beginning parameter. The parameters define the range of nodes, modes, etc., to process. *(Note: AMV+ analyses can be very time consuming. The user should be aware of the number of probabilistic analyses being specified).*

Summary of *AMVDEFINE keywords

- *AMVDEFINE** Signals beginning of advanced mean value keywords.
- *ITER** Defines the convergence criteria for the AMV+ p-level and z-level algorithms.
- *CONDITIONNUMBER** Selects the condition (increment, modal or spectral) numbers.
- *NODENUMBER** Selects the node numbers.
- *COMPONENTNUMBER** Selects the component numbers.
- *LAYERNUMBER** Selects the plate element layer numbers.
- *END** Signals end of mean value keywords.

The purpose of these keywords is identical to version 4.2. The ***ITER** keyword is a new keyword.

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***AMVDEFINE**

***AMVDEFINE** signal the start of the mean value keywords. The following keywords will be interpreted as advanced mean value keywords until the ***END** keyword is reached.

*COND	icond1	icond2
--------------	---------------	---------------

***COND** selects the beginning and ending condition numbers for the mean value analysis. Condition refers to either incremental, modal, or harmonic/spectral analysis. The condition type is selected with the ***DATATYPE** keyword in the **MVDEFINE** section. NESSUS will perform AMV+ analyses from condition number **icond1** to **icond2**.

icond1 = beginning condition number for MV analyses.

icond2 = ending condition number for MV analyses.

icond1 default = 1.

icond2 default = **icond1**.

*NODE	inode1	inode2
--------------	---------------	---------------

***NODE** selects the beginning and ending node numbers for the advanced mean value analysis. NESSUS will perform probabilistic analysis from node number **icond1** to **icond2**.

inode1 = beginning node number for MV analyses.

inode2 = ending node number for MV analyses.

inode1 default = 1.

inode2 default = **inode1**.

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*COMP	icomp1	icomp2
--------------	---------------	---------------

***COMP** selects the beginning and ending component numbers for the advanced mean value analysis. NESSUS will perform probabilistic analysis from component number **icomp1** to **icomp2**. Allowable component numbers are 0 through 6.

icomp1 = beginning component number for MV analyses.

icomp2 = ending component number for MV analyses.

icomp1 default = 1.

icomp2 default = **icomp1**.

*LAYER	ilay1	ilay2
---------------	--------------	--------------

***LAYER** selects the beginning and ending layer numbers for the advanced mean value analysis of the plate element (#75). NESSUS will perform probabilistic analysis from layer number **ilay1** to **ilay2**. Allowable layer numbers are 1 through 5.

ilay1 = beginning layer number for MV analyses.

ilay2 = ending layer number for MV analyses.

ilay1 default = 1.

ilay2 default = **ilay1**.

*ITER
itmax tolerr

***ITER** defines the convergence criteria for the AMV+ z-level and p-level iteration procedures. The probabilistic algorithm will continue until either the maximum number of iterations is reached, **itmax**, or the convergence tolerance, **tolerr**, is satisfied. The default is AMV analysis, **itmax** = 1 and **tolerr** = 1.E-8.

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***END**

END signals the end of input for the *AMVDEFINE section.

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4.0 CVARIABLE Transformations and Response Values

The following sections present valid transformation options, response type options, and component definitions for each response quantity available.

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4.1 Valid Transformations

TRANSFORMATION jtran
data block if jtran =1

where

jtran = transformation option

and

jtran = 0 => no transformation will
be performed

jtran = 1 => quantities will be rotated to a user-defined
coordinate system as described below

jtran = 2 => quantities will be rotated to the principal
value coordinate system

DATA BLOCK FOR jtran =1 FOR ELEMENT TYPES 7 AND 154

TRANSFORMATION 1
jaxis(1) ang(1)
jaxis(2) ang(2)
jaxis(3) ang(3)

where

jaxis = axes of rotation
ang = angles of rotation in degrees

and

jaxis = 1 => x-axis
jaxis = 2 => y-axis
jaxis = 3 => z-axis

The user-defined coordinate system $x^3y^3z^3$ is defined in terms of the global coordinate system $x^0y^0z^0$ as follows. First, $x^1y^1z^1$ is constructed by rotating $x^0y^0z^0$ by $\text{ang}(1)$ about the positive $\text{jaxis}(1)$ -axis of $x^0y^0z^0$ in the right-handed sense. Second, $x^2y^2z^2$ is constructed by rotating $x^1y^1z^1$ by $\text{ang}(2)$ about the positive $\text{jaxis}(2)$ -axis of $x^1y^1z^1$ in the right-handed sense. Third and last, $x^3y^3z^3$

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is constructed by rotating $x^2y^2z^2$ by $\text{ang}(3)$ about the positive $\text{jaxis}(3)$ -axis of $x^2y^2z^2$ in the right-handed sense. From this procedure, in three dimensions, any coordinate system $x^3y^3z^3$ can be constructed from $x^0y^0z^0$.

DATA BLOCK FOR $\text{jtran} = 1$ FOR ELEMENT TYPES 10 AND 153

TRANSFORMATION 1

idummy angle
idummy rdummy
idummy rdummy

where

idummy = unused integer value
rdummy = unused real value
angle = angle of rotation in degrees

The user-defined coordinate system z^1r^1 is defined in terms of the global coordinate system z^0r^0 as depicted in Figure 2.

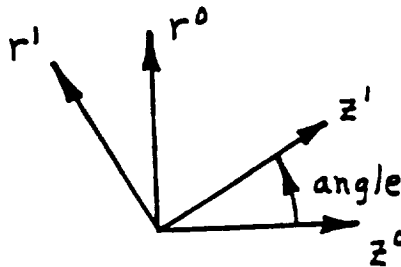


Figure 2.

DATA BLOCK FOR $\text{jtran} = 1$ FOR ELEMENT TYPES 3, 11, 151 AND 152

TRANSFORMATION 1

idummy angle
idummy rdummy
idummy rdummy

where

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idummy = unused integer value
rdummy = unused real value
angle = angle of rotation in degrees

The user-defined coordinate system x^1y^1 is defined in terms of the global coordinate system x^0y^0 of the perturbation database, as depicted in Figure 3.

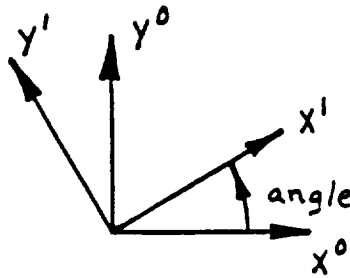


Figure 3.

DATA BLOCK FOR jtran = 1 FOR ELEMENT TYPES 75 AND 98

The option jtran = 1 is not currently implemented for these elements since mesh data (which may not reside on the perturbation database) are needed to calculate rotations from the global coordinate system.

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4.2 Response Values for Incremental Analysis

Table 1. Admissible values of jresp (response type) for incremental problems.

1 = total displacement
2 = total strain
3 = total stress
11 = plastic strain
12 = backstress
13 = creep strain
14 = thermal strain
17 = generalized strain
18 = generalized stress
21 = material state variables
25 = velocities
26 = accelerations

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Table 2. Admissible values of jtran (transformation option) for each incremental response type (jresp) and element type (NA = response type not available).

	3	7	10	11	75	98	151	152	153	154	element type
1	0 1	0 1	0 1	0 1	0	0	0 1	0 1	0 1	0 1	
2	0 1 2	0 1 2	0 1 2	0 1 2	0	NA	0 1 2	0 1 2	0 1 2	0 1 2	
3	0 1 2	0 1 2	0 1 2	0 1 2	0	NA	0 1 2	0 1 2	0 1 2	0 1 2	
11	0 1 2	0 1 2	0 1 2	0 1 2	0	NA	0 1 2	0 1 2	0 1 2	0 1 2	
12	0 1 2	0 1 2	0 1 2	0 1 2	0	NA	0 1 2	0 1 2	0 1 2	0 1 2	
13	0 1 2	0 1 2	0 1 2	0 1 2	0	NA	0 1 2	0 1 2	0 1 2	0 1 2	
14	0 1 2	0 1 2	0 1 2	0 1 2	0	NA	0 1 2	0 1 2	0 1 2	0 1 2	
17	NA	NA	NA	NA	0	0	NA	NA	NA	NA	
18	NA	NA	NA	NA	0	0	NA	NA	NA	NA	
21	0	0	0	0	0	NA	0	0	0	0	
jresp											

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25	0 1	0 1	0 1	0 1	0	0	0 1	0 1	0 1	0 1
26	0 1	0 1	0 1	0 1	0	0	0 1	0 1	0 1	0 1
jresp										

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Available components for incremental problems

1) Component definitions for scalar response quantities

jresp = 21: (material state variables)

component 1 = equivalent plastic strain

component 2 = equivalent creep strain

component 3 = equivalent swell strain

2) If jtran = 0 (no transformation), then

All usual components are available (including component #0 if it happens to be defined).

3) If jtran = 1 (rotation), then

Component #0 is available (if it happens to be defined).

Other components (1 , 2 , ... , etc.) are available in the new user-defined coordinate system.

4) If jtran = 2 (principal values), then the available components are

0 = effective stress or effective strain

1 = minimum principal stress or principal strain

2 = intermediate principal stress or principal strain

3 = maximum principal stress or principal strain

4 = maximum shear stress or maximum engineering shear strain

except for element types 3 and 151 when jresp = 2, 11, 13, or 14, in which case the available components are

0 = effective strain

1 = minimum principal strain in the xy-plane

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2 = returned as zero

3 = maximum principal strain in the xy-plane

4 = maximum engineering shear strain in the xy-plane

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4.3 Response Values for Eigenvalue Analysis

Table 3. Admissible values of jresp (response type) for eigen-problems.

- 30 = the eigenvalue for the mode
- 31 = modal displacement (eigenvector)
- 32 = modal strain
- 33 = modal stress
- 35 = the frequency in radian per time
- 36 = the frequency in cycles per time

DRAFT

Table 4. Admissible values of jtran (transformation option) for each eigen-problem response type (jresp) and element type (NA = response type not available).

	3	7	10	11	75	98	151	152	153	154	element type
30	0	0	0	0	0	0	0	0	0	0	
31	0 1	0 1	0 1	0 1	0	0	0 1	0 1	0 1	0 1	
32	0 1 2	0 1 2	0 1 2	0 1 2	0	NA	0 1 2	0 1 2	0 1 2	0 1 2	
33	0 1 2	0 1 2	0 1 2	0 1 2	0	NA	0 1 2	0 1 2	0 1 2	0 1 2	
35	0	0	0	0	0	0	0	0	0	0	
36	0	0	0	0	0	0	0	0	0	0	
jresp											

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Available components for eigen-problems

1) Component definitions for scalar response quantities

jresp = 30:

component 1 = the eigenvalue for the mode

jresp = 35:

component 1 = the frequency in radian per time

jresp = 36:

component 1 = the frequency in cycles per time

2) If jtran = 0 (no transformation), then

All usual components are available (including component #0 if it happens to be defined).

3) If jtran = 1 (rotation), then

Component #0 is available (if it happens to be defined).

Other components (1 , 2 , ... , etc.) are available in the new user-defined coordinate system.

4) If jtran = 2 (principal values), then the available components are

0 = effective stress or effective strain

1 = minimum principal stress or principal strain

2 = intermediate principal stress or principal strain

3 = maximum principal stress or principal strain

4 = maximum shear stress or maximum engineering shear strain

except for element types 3 and 151 when jresp = 32, in which case the available components are

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- 0 = effective strain
- 1 = minimum principal strain in the xy-plane
- 2 = returned as zero
- 3 = maximum principal strain in the xy-plane
- 4 = maximum engineering shear strain in the xy-plane

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4.4 Response Values for Spectral Analysis

Table 5. Admissible values of jresp (response type) for spectral problems.

51 = real displacement
52 = real strain
53 = real stress
61 = imaginary displacement
62 = imaginary strain
63 = imaginary stress
71 = the amplitude of the displacement
72 = the amplitude of the strain
73 = the amplitude of the stress
81 = the phase of the displacement
82 = the phase of the strain
83 = the phase of the stress
91 = mean square displacement
92 = mean square strain
93 = mean square stress
96 = stress velocity -

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Table 6 (1 of 2). Admissible values of jtran (transformation option) for each spectral response type (jresp) and element type (NA = response type not available).

	3	7	10	11	75	98	151	152	153	154	element type
51	0 1	0 1	0 1	0 1	0	0	0 1	0 1	0 1	0 1	
52	0 1 2	0 1 2	0 1 2	0 1 2	0	NA	0 1 2	0 1 2	0 1 2	0 1 2	
53	0 1 2	0 1 2	0 1 2	0 1 2	0	NA	0 1 2	0 1 2	0 1 2	0 1 2	
61	0 1	0 1	0 1	0 1	0	0	0 1	0 1	0 1	0 1	
62	0 1 2	0 1 2	0 1 2	0 1 2	0	NA	0 1 2	0 1 2	0 1 2	0 1 2	
63	0 1 2	0 1 2	0 1 2	0 1 2	0	NA	0 1 2	0 1 2	0 1 2	0 1 2	
71	0	0	0	0	0	0	0	0	0	0	
72	0	0	0	0	0	NA	0	0	0	0	
jresp											

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Table 6 (2 of 2). Admissible values of jtran (transformation option) for each spectral response type (jresp) and element type (NA = response type not available).

	3	7	10	11	75	98	151	152	153	154	element type
73	0	0	0	0	0	NA	0	0	0	0	
81	0	0	0	0	0	0	0	0	0	0	
82	0	0	0	0	0	NA	0	0	0	0	
83	0	0	0	0	0	NA	0	0	0	0	
91	0	0	0	0	0	0	0	0	0	0	
92	0	0	0	0	0	NA	0	0	0	0	
93	0	0	0	0	0	NA	0	0	0	0	
96	0 1 2	0 1 2	0 1 2	0 1 2	0	NA	0 1 2	0 1 2	0 1 2	0 1 2	
jresp											

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Available components for spectral problems

- 1) If $jtran = 0$ (no transformation), then

All usual components are available (including component #0 if it happens to be defined)

except

component #0 is not available for $jresp = 72, 73, 82, 83, 92$ or 93 .

- 2) If $jtran = 1$ (rotation), then

Component #0 is available (if it happens to be defined).

Other components (1 , 2 , ... , etc.) are available in the new user-defined coordinate system.

- 3) If $jtran = 2$ (principal values), then the available components are

0 = effective stress or effective strain

1 = minimum principal stress or principal strain

2 = intermediate principal stress or principal strain -

3 = maximum principal stress or principal strain

4 = maximum shear stress or maximum engineering shear strain

except for element types 3 and 151 when $jresp = 52$ or 62 , in which case the available components are

0 = effective strain

1 = minimum principal strain in the xy-plane

2 = returned as zero

3 = maximum principal strain in the xy-plane

4 = maximum engineering shear strain in the xy-plane

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5.0 NESSUS/RISK

A risk module has been added to NESSUS that will compute the risk with respect to cost, performance, and a user-defined criteria.

COST

Risk with respect to cost is computed using the formula

$$\text{Risk/Cost} = C_0(x) + P_f(x) * C(x) + (1-P_f)*NC(x)$$

where x is the allowable response in a reliability analysis such as stress, displacement, cycles to failure, C_0 is the initial cost as a function of x , P_f is the probability of failure as a function of x , and $NC(x)$ is the probability of nonfailure as a function of x . The $P_f(x)$ curve may be the CDF computed by NESSUS; however, in general, x can be any design parameter including non-random variables.

The four functions, $P_f(x)$, $C_0(x)$, $C(x)$ and $NC(x)$ are defined with appropriate keywords defined below. The failure function, $P_f(x)$, is defined in table form using the ***XPF** keyword. The three cost functions, $C_0(x)$, $C(x)$, and $NC(x)$ are defined using the ***CRUNC** keyword. The XPF and CFUNC x values do not have to coincide. The output points, x_i , at which the risk is to be computed are defined with the ***XCOUT** keyword. The output points do not have to coincide with the failure function or cost function points. RISK will interpolate linearly the failure function and cost function values to use in the equation for risk. However, no extrapolation is allowed; therefore, the output points must lie within the failure function and cost function data.

Example:

The user has computed a P_f function which is the probability of failure of a structure as a function of a member area, A . Thus, x is the member area A . (Note, in this example, x is not a PFEM response.) The initial costs are assumed to vary linearly with A . The cost of a failure is assumed to be independent of A ; thus, the cost function, or consequences function, is assumed to be constant with A . The cost of nonfailure is also assumed to be constant with A , although at a far less magnitude than the cost of failure function. The user desires the risk at a number of points between the areas 0.1 and 1.0.

The input file would look like:

```
*RISK          to run risk stand-alone, start file with *RISK
C              comments are allowed
C
C              define the probability of failure function
C              five points will be used
```

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```
*XPF
  *TABL          5    indenting is allowed
0.05             .999
0.3              .980
0.5              .950
0.7              .925
1.0              .9
*END
C
C    define cost functions
C    three points will be used
*COST
*CFUNC          3
0.1  2.0  50.  1.0
0.5  3.11      50.1.0
1.0  4.0  50.  1.0
C
C    define the output points
C    (areas for which risk is to be computed)
C    ten points are used
*XCOUT          10
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
*END
C    end cost input
*END
```

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PERFORMANCE

A second analysis method in RISK is based on performance. In this approach, the risk with respect to performance is adjusted by changing the primitive random variables to meet a user-specified probability of failure at user specified design values x . This is, in essence, part of the design process.

The input requirements are the P_f function which can be computed automatically by PFEM, the random variables statistics and probabilistic sensitivities, and the design requirements. The P_f function is input with the ***XPF** keyword as before. The random variable statistics and probabilistic sensitivities are input with the ***RVDEFINE** keyword. The design requirements are input under the ***PERFORMANCE** keyword by using the ***DREQ** and ***DVAR** keywords. NESSUS/RISK will compute the necessary changes in the random variable means and standard deviations needed to meet the requirements. Each random variable will be adjusted independent of the others, and the mean and standard deviation are adjusted independently.

The algorithms used in RISK are described in the OCT. 1991 PSAM monthly, Appendix C.

Assumptions:

The user should be aware of the assumptions involved in the implemented algorithms. First, for nonlinear functions and nonnormal distributions, the computed design changes are only a first order estimate. An iterative process is needed for an accurate solution, as outlined in the technical writeup. Second, the probabilistic sensitivities are assumed constant, whereas, in general they will vary over the range of the failure function. It is assumed that when utilizing RISK/Performance, the user will be focusing on a specific area of the failure function, e.g., left tail, and the probabilistic sensitivities will be approximately constant. Third, the technical writeup details an algorithm for a variety of nonnormal distributions. In this version of NESSUS/RISK, a normal distribution is assumed for all random variables.

Example:

This problem has two random variables, x_1, x_2 with statistics

$\mu_1 = 25.$ $\sigma_1 = 4.$ normal distribution

$\mu_2 = 10.$ $\sigma_2 = 3.$ normal distribution

and probabilistic sensitivity factors

$\alpha_1 = -0.8$ $\alpha_2 = 0.6$

The failure function is defined by two points in the left tail.

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$x = 1$ $P_f = 1.35E-3$
 $x = 2$ $P_f = 0.1$

The design requirements are:

$P_f = 2.33E-4$ at $x = 1$

and

$P_f = .001$ at $x = 1.5$

Both the mean and standard deviation will be changed independently to meet the design requirements if needed.

The input file would look like:

```
*RISK
*PERFORMANCE
  *DREQ          2          two points are requested
    1.0  2.33E-4
    2.0  0.001
C
C              -
C              both mean and sigma to be varied
  *DVAR          BOTH
*END
C
*XPF
  *TABLE          2          two points used
    1.0          1.35E-3
    2.0          0.1
*END
C
C      define random variables
C
C      note: presently all ran vars assumed normal
*RVDEFINE
  *DEFINE      1
    25.  4.  NORMAL
    -0.8
  *DEFINE      2
    10.  3.  WEIBULL
    0.6
*END
C
C      end risk input
*END
USER
```

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An option for a user-defined criteria for risk is provided through a user-written subroutine, USRRSK. The P_r function defined by *XPF and the random variable statistics defined through *RVDEFINE can be used. This option can be used when running risk stand-alone or with PFEM. A copy of USRRSK is given below.

```

SUBROUTINE USRRSK(IERROR)
  IMPLICIT REAL*8 (A-H,O-Z)

C
  PARAMETER (MRANV=100)
  PARAMETER (MAXNP = 100)

C
C*****
C **                USER-DEFINED RISK                **
C **  THIS SUBROUTINE IS ACTIVATED BY THE *USER KEYWORD IN A  **
C **  NESSUS/RISK INPUT FILE. THE *XPF AND *RVDEFINE INFORMATION **
C **  CAN BE USED WITH THIS OPTION.                      **
C*****
C-----
C
  COMMON / ALGEM / ICREAD,ILPRNT,JLPRNT,ICONSL,IPOSTF,ISCRAF,
1          IPLOTB,IRSTRT,JCREAD,IRVBIN,IDBASE,IRVDEF,
2          PIDUM,LINE,LINE2

C
C          random variable definitions
COMMON /RVDEFI/ NUMRVS,LRVNUM(MRANV),RVMEAN(MRANV),RVSDEV(MRANV),
+          LRV DST(MRANV),RVLLMT(MRANV),RVULMT(MRANV),
+          LRVNMK(MRANV),RVCOEF(9,MRANV)

C
COMMON /RVSENS/ ALPHA
DIMENSION ALPHA(MRANV)

C
C          probability of failure input
COMMON / XPFCN / XPF,NXPF,XPNEW

C
C-----
  DIMENSION XPF(MAXNP,2),XPNEW(MAXNP)
C*****
C          variable description
C*****
C          ICREAD - input unit to read from
C          ILCONSL - write to screen
C          ILPRNT - output file
C
C          XPF(I,J) - failure function
C                   - row I is x value

```

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```

C      - col j is pf value
C      NXPF - number of XPF points
C
C      -----
C      DESCRIPTION OF COMMONS: /RFDEFI/ AND /RVNAME/
C      -----
C      HOLDS THE INFORMATION READ IN FROM THE *DEFINE INPUT DATA
C
C      /RFDEFI/ NUMRVS - NUMBER OF RANDOM VARIABLES
C      /RFDEFI/ LRVNUM - RANDOM VARIABLE NUMBER
C      /RFDEFI/ RVMEAN - MEAN
C      /RFDEFI/ RVSDEV - STANDARD DEVIATION
C      /RFDEFI/ LRV DST - DISTRIBUTION
C      /RFDEFI/ RVLLMT - LOWER LIMIT (ONLY FOR LRV DST = 6,9,10)
C      /RFDEFI/ RVULMT - UPPER LIMIT (ONLY FOR LRV DST = 6,9,10)
C      /RFDEFI/ LRVNMK - NUMBER OF COEFFICIENTS (ONLY FOR LRV DST = 6,7)
C      /RFDEFI/ RVCOEF - COEFFICIENTS
C      /RVNAME/ RVNAME - RANDOM VARIABLE NAME
C
C      ALPHA(I) - prob. sensitivities of random variables I
C
C      IERROR = ERROR INDICATOR
C      IERROR = 0
C      *****
C      -----
C
C      USER-DEFINED RISK
C
C      -----
C
C      RETURN
C      END

```

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Executing Risk

NESSUS/RISK can be executed stand-alone or in an automated analysis with PFEM.

RISK Stand-alone

Begin input file with *RISK keyword. All input must be in input file.

RISK with PFEM

In this instance, the risk analysis will be performed after a PFEM analysis. The RISK input must appear at the end of the usual PFEM input file. A schematic of the input file is given below.

```
*PFEM
      PFEM input data
*END
C-----
*FEM
      FEM input data
*END
C-----
*FPI
      FPI input data
*END
C-----
*RISK
      RISK input data
*END
```

When coupled with PFEM, the RISK P_r function defined by *~~xx~~ will be the CDF computed by PFEM. All other data must be supplied in the RISK input data section. In procedural terms, the steps that will be taken are:

- 1) PFEM will compute the CDF using the AMV+ algorithm, etc., exactly as before.
- 2) PFEM will write a RISK input file and insert the computed CDF under the *~~xx~~ keyword.
- 3) RISK will be called to compute the risk with respect to the user requirements.
- 4) PFEM will proceed to the next node, increment, mode, etc., and steps 1-3 will be repeated according to the input file.

All of this is fully automated from a *single* input file. The risk output will appear in the jobname.out file.

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RISK Input Format

RISK will compute the risk with respect to cost, performance, and a user-defined criteria. Currently, risk can be computed for only one of the analysis types in a single analysis.

There are currently six major keyword sections in RISK. They are: ***RISK** which must precede all other risk keywords, ***COST**, ***PERFORMANCE**, ***USER**, ***XPF**, ***RVDEFINE**. ***RISK** must precede these keywords. As with other NESSUS input modules, only the first 4 characters are significant; the input is column independent and the order is unimportant.

Summary of RISK keywords

- *XPF** - indicates the beginning of the failure function definition.
- *RVDEFINE** - indicates the beginning of the random variable definitions and sensitivities.
- *COST** - signifies the risk cost algorithm is to be used.
- *PERFORMANCE** - signifies the risk performance algorithm is to be used.
- *USER** - signifies a user-defined risk algorithm is to be used.
- *END** - ends the risk input datafile.

A schematic of the risk keywords is given below.

***RISK**

***XPF**

***TABLE**

***END**

***RVDEFINE**

***DEFINE**

***END**

***COST**

***CFUNCTION**

***XCOUT**

***END**

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```
*PERFORMANCE
  *DREQ
  *DVAR
*END
```

```
*USER
*END
```

```
*END
```

```
*XPF
```

***XPF** signals the start of the definition of the failure function. This function is defined in terms of a variable x and a probability of failure. This function can of course represent a CDF computed from NESSUS but need not. The XPF data can be used with any risk analysis type.

***XPF** itself has subkeywords used to designate the input format. Currently, NESSUS 5.0 is limited to tabular input only. The subkeywords for XPF are ***TABLE** and ***END**. ***TABLE** is used to signify tabular input and ***END** is used to signify the end of the XPF data.

```
*TABLE          npts
```

Signifies that the input is to be in tabular form.

`npts` signifies the number of points in the table.

```
*END
```

Signifies the end of the XPF data.

For example, an XPF curve defined by 5 points would be input as

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```
*XPF
*TABL      5
X1         PF1
X2         PF2
X3         PF3
X4         PF4
X5         PF5
*END
```

where x's are the variable points and the P_i's are the corresponding probabilities of failure. As always, the keyword section is closed with a ***END**.

```
-----
*RVDEFINE
-----
```

This section is used to define the mean, standard deviation, distribution type, and probabilistic sensitivity for each random variable. This information is necessary for the risk performance algorithm, but can be used by any analysis method. The input format is similar to that used in the RVDEFINE section of PFEM.

The subkeywords for the RVDEFINE section are: ***DEFINE** and ***END**.

```
-----
*DEFINE      irvnum
mean      stdev      dist
alpha
-----
```

This keyword signals the definition of the random variable mean, standard deviation, distribution type, and probabilistic sensitivity factor for random variables, irvnum.

irvnum is the random variable number.

mean is the random variable mean value.

stdev is the random variable standard deviation.

dist is the random variable distribution type.

alpha is the random variable probabilistic sensitivity factor.

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***DEFINE** is repeated **numrv** times, where **numrv** is the number of random variables.

***END**

Signifies the end of the RVDEFINE data.

As an example, for a problem with 2 random variables, the RVDEFINE section might look like

```
*RVDEFINE
*DEFINE      1
25.      4.      NORMAL
-0.8
*DEFINE      2
10.      3.      WEIBULL
0.6
*END
```

***COST**

This section signals that the cost algorithm will be used. The subkeywords for the COST section are: ***CFUNCTION** and ***XCOUT** and ***END**.

***CFUNCTION** **ncpts**
x_i **c0_i** **c_i** **nc_i** **i=1,ncpts**

This keyword signals the input of the cost function input.

ncpts is the number of cost function entries.

The input is in four columns: **x** value, initial cost at this **x**, cost at this **x**, and noncost at this **x**.

x_i is the **x** value point.

c0_i is the initial cost at **x_i**.

c_i is the cost of failure at **x_i**.

nc_i is the cost of nonfailure at **x_i**.

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```
-----  
*XCOUT      nxcpts  
xi          i=1, nxcpts  
-----
```

This section signals the definition of the user-defined output points.

nxcpts is the number of output points.

x_i is the x value of the output points.

```
-----  
*END  
-----
```

Signifies the end of the COST data.

```
-----  
*PERFORMANCE  
-----
```

This section defines the keywords for computing the risk with respect to performance. The subkeywords for the COST section are: *DVAR and *DREQ and *END.

```
-----  
*DREQ          ndrpts  
xi            Pfi          i=1, ndrpts  
-----
```

This keyword is used to input the design requirements which consist of required probability of failure at a design value x.

ndrpts is the number of design requirement input points.

x_i is the design value.

Pf_i is the required probability of failure at x_i.

For example, if the requirements are a probability of failure of 0.001 at x=10, and P_f = 0.002 at x=20, the input would be:

```
*DREQ          2  
10.            0.001  
20.            0.002
```

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***DVAR** **desvar**

This keyword indicates which variable, (mean, or standard deviation, or both) to alter to meet the requirements. The mean or standard deviation is modified independently.

desvar = 'MEAN', 'STDEV', or 'BOTH'

***END**

Signifies the end of the PERFORMANCE data.

***USER**

This keyword signifies that the user-defined subroutine USRRSK will be called. The XPF and RVDEFINE data sections *can* be used with USER.

***END**

Signifies the end of the USER data.

***END**

Signifies the end of the entire risk input file.

APPENDIX B
Resistance Model Research

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Appendix B.1
A Review of Fatigue Models

University of Arizona

A REVIEW OF FATIGUE MODELS

The limit state expressions and reliability analysis strategies relative to the models.

by

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STRESS BASED FATIGUE LIFE PREDICTION

C-1 The characteristic S-N curve; constant amplitude stress; no endurance limit, mean stress; notch effects; multiaxial stress.

C-2 Multi-linear S-N curve (includes endurance limit); constant amplitude stress

C-3 The modified Goodman model; constant amplitude stress; includes both fatigue and yield failure modes.

C-4 The general S-N curve (non-linear) representation of fatigue strength, e.g., MIL-HDBK-5E; constant and variable amplitude stress; mean stress.

C-5 Characteristic S-N curve with no endurance limit; variable amplitude stresses; linear damage accumulation; general variable amplitude stress; stationary Gaussian processes including both narrow band and wide band; mean stress.

C-6 Characteristic S-N curve with no endurance limit; variable amplitude stresses including sequence effects; general variable amplitude stress; stationary Gaussian processes including both narrow band and

wide band; mean stress.

C-7 Multilinear S-N curve; variable amplitude stresses; no sequence effects

STRAIN-BASED FATIGUE LIFE PREDICTION (Local Strain Analysis)

L-1 The high cycle component of the general strain life relationship; constant amplitude stress; mean stress; notch effects; stress endurance

L-2 The general strain-life relationship and low cycle fatigue; constant amplitude stress; mean stress; stress relaxation

L-3 Low cycle fatigue under random stresses; no sequence effects; mean stress; stress relaxation

THE FRACTURE MECHANICS MODEL TO DESCRIBE FATIGUE CRACK PROPAGATION (subcritical crack growth)

F-1 Paris law to describe fatigue crack growth; no threshold level; constant amplitude stresses; mean stress.

F-2 Paris law to describe fatigue crack growth; no threshold level; variable amplitude stresses including stationary random processes; equivalent stresses

F-3 Paris law to describe fatigue crack growth; threshold stress intensity factor; variable amplitude stresses including stationary random processes; equivalent stresses

F-4 Polynomial models to describe fatigue crack growth; variable amplitude stresses; equivalent stresses

F-5 Crack growth models for variable amplitude stresses where

sequence effects are important; computer codes for cycle-by-cycle counting.

HIGH TEMPERATURE, LOW CYCLE FATIGUE (synergistic effects of creep and fatigue)

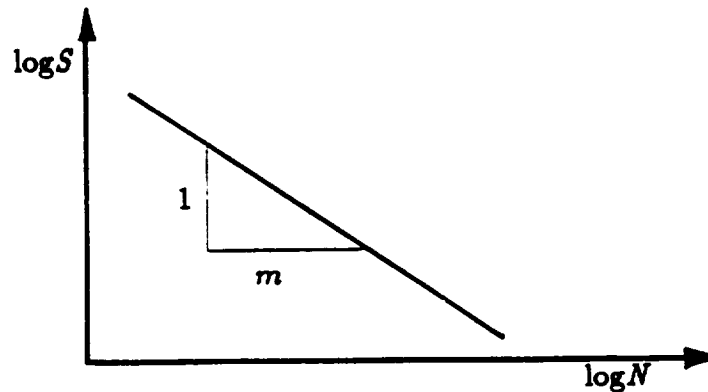
H-1 Linear and elliptical rule; constant amplitude stress

H-2 Frequency modified strain; constant amplitude stress

H-4 Strain range partitioning; constant amplitude stress

C-1 Characteristic S - N Curve High Cycle Fatigue (Constant Amplitude Stress)

- Model for strength



Cycles to failure

$$N = AS^{-m}$$

Parameters :

- Applied stress range, S
- Fatigue strength coefficient, A
- Fatigue strength exponent, m

- Assumptions and Comments

1. Cycle life $N > 10^5$.
2. Smooth surface.(no cracks)
3. No aggressive corrosion.
4. No creep. (high temperature effects)
5. No endurance limit.
6. Constant amplitude stress.

- Uniaxial Stresses

σ can be stress range σ_r or stress amplitude σ_a

Mean stress models

Soderberg's linear model	$\frac{\sigma_a}{S_N} + \frac{\sigma_m}{S_y} = 1$
Goodman's linear model	$\frac{\sigma_a}{S_N} + \frac{\sigma_m}{S_u} = 1$
Morrow's linear model	$\frac{\sigma_a}{S_N} + \frac{\sigma_m}{\sigma_f} = 1$
Gerber's parabolic model	$\frac{\sigma_a}{S_N} + \left(\frac{\sigma_m}{S_u}\right)^2 = 1$
Elliptic model	$\left(\frac{\sigma_a}{S_N}\right)^2 + \left(\frac{\sigma_m}{S_u}\right)^2 = 1$

where

σ_m : mean stress

S_u : ultimate strength

S_N : fatigue strength at N cycle

S_y : yielding strength

- Multiaxial Stresses ("a" or "m" indicates amplitude or mean of the stress component).

$$\overline{\sigma_a} = \sqrt{\frac{1}{2}[(\sigma_{xa} - \sigma_{ya})^2 + (\sigma_{ya} - \sigma_{za})^2 + (\sigma_{za} - \sigma_{xa})^2 + 6(\tau_{xya}^2 + \tau_{yza}^2 + \tau_{zxa}^2)]}$$

$$\overline{\sigma_m} = \sigma_{xm} + \sigma_{ym} + \sigma_{zm}$$

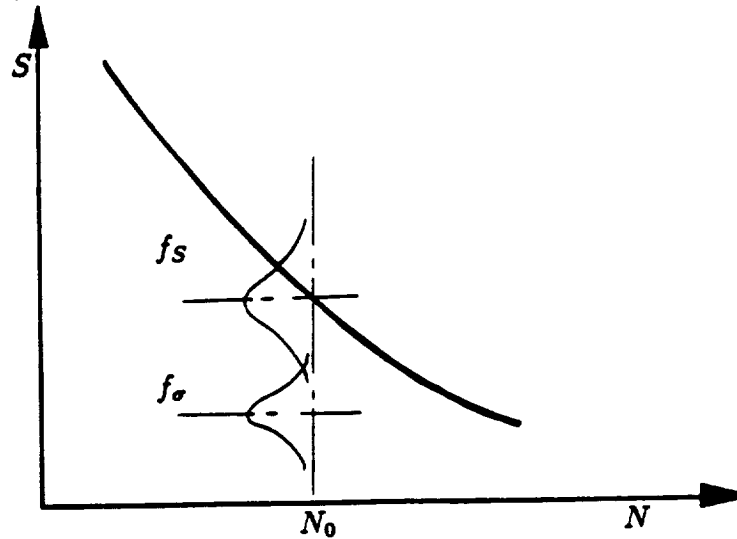
$$\sigma_{ar} = \frac{\overline{\sigma_a}}{1 - \frac{\overline{\sigma_m}}{S_u}} \quad \text{equivalent completely reversed uniaxial stress}$$

- Reliability Considerations :

let $m = \text{constant}$, $A = \text{a random variable describing the variability in material behavior.}$

- a.) Failure when $(N < N_0) = (A < N_0 S^m)$; where N_0 is design life; and N is cycles to failure, a random variable (maybe difficult to formulate in some cases). Probability of failure is $P_f = P_r[A < N_0 S^m]$

b.) Failure when ($S < \sigma$) at given N_0



$$P_f = P_r[\sigma > S]$$

$$= P_r[\sigma > (\frac{A}{N_0})^{\frac{1}{m}}]$$

Note : Here S and A are random variables. Typically $C_S \approx 10\%$

• References

- [1] Collins, J. A., Failure of Materials in Mechanical Design, John Wiley & Sons, 1981.
- [2] Shigley, J. E., Mechanical Engineering Design, McGraw-Hill, 1972.
- [3] Dowling, N. E., "Deformation, Fracture, and Fatigue of Engineering Materials," Virginia Polytechnical U., 1989. (manuscript in development)

C-2 Multi-Linear $S - N$ Curve (Under Constant Amplitude Stress)

- Basic Assumptions:

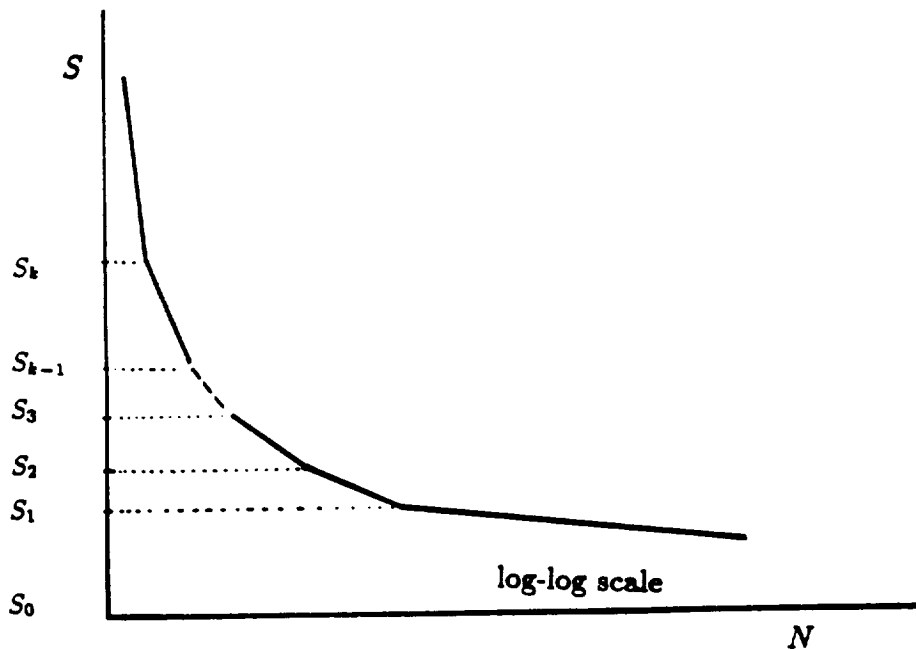
1. Stress is constant amplitude having amplitude (or range), σ . The mean stress is σ_m .
2. Stress modeling error is given by the random variable B so that the actual stress is a random variable,

$$\sigma = B\sigma'$$

where σ' is the estimate of stress, S , using the best predictive method.

3. The service life is given as N_0 .
4. Fatigue strength is defined by a multilinear $S - N$ curve.

- Model for fatigue strength



$$NS^{m_1} = A_1, \quad \text{for } S_0 \leq S < S_1$$

$$NS^{m_2} = A_2, \quad \text{for } S_1 \leq S < S_2$$

$$NS^{m_3} = A_3, \quad \text{for } S_2 \leq S < S_3$$

.

.

$$NS^{m_k} = A_k, \quad \text{for } S_{k-1} \leq S < S_k$$

or, in general form,

$$NS^{m_i} = A_i, \quad \text{for } S_{i-1} \leq S < S_i \quad i = 1, 2, 3, \dots, k$$

where

S : applied stress range or amplitude

A_i : fatigue strength coefficient

m_i : fatigue strength exponent

Let $S_0 = 0$ and $S_\infty = \infty$.

• Mean Stress Effects:

If $\sigma_m > 0$, then the fatigue strength should be

$$A_i = A_{0i} \left[1 - \frac{\sigma_m}{S_u} \right]^{m_i}$$

where A_{0i} is the fatigue strength coefficient for the i th segment for zero mean stress.

• Reliability Consideration:

1. The event of failure prior to completion of the service life N_0 is $[\sigma < S]$
2. The probability of failure is

$$P_f = \Pr[\sigma < S]$$

Reliability prediction can be made by Wu/FPI or AMVFO.

• Comments:

1. Existence of endurance limit.

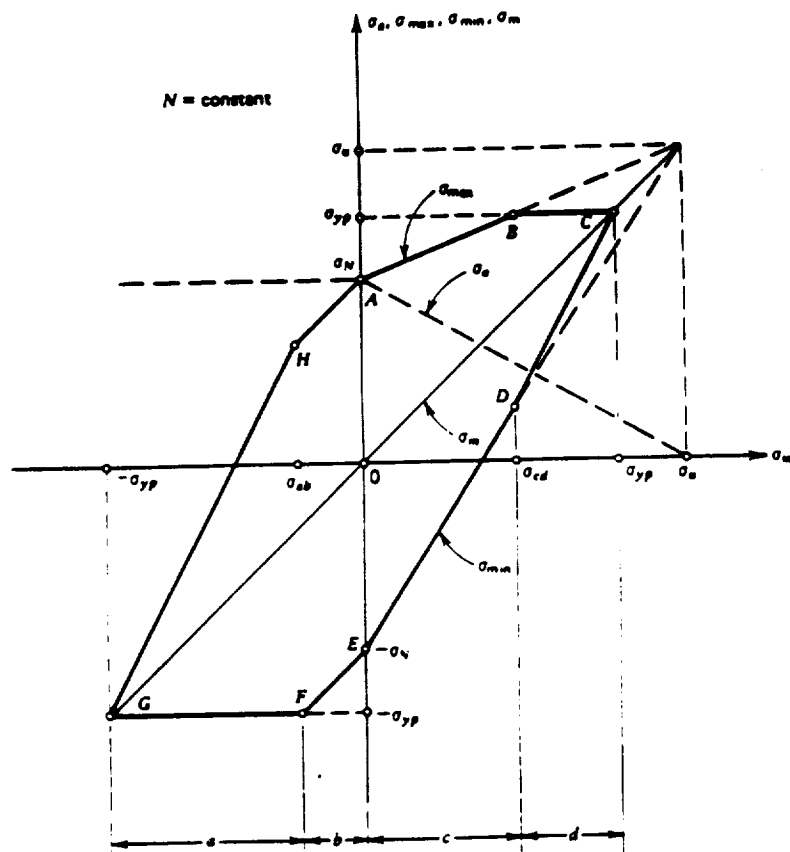
If an endurance limit exists, then the linear section representing the endurance limit is in the form of $NS^{m_1} = A_1$ with $A_1 = \infty$. Thus the first term of Eq. 1. vanishes. Eq. 1. will not lose its generality with the existence of the endurance limit.

• References

- [1] Wirsching, P. H. and Light, M. C., "Fatigue Under Wide Band Random Stresses," Journal of the Structural Division, ASCE, Vol. 108, No. ST7, pp.1593-1607, 1979.
- [2] Wirsching, P. H., "Considerations of Probability Based Fatigue Design for Marine Structures," Marine Structure, 1 (1988) 23-45.
- [3] Wirsching, P. H., "Fatigue Reliability for Offshore Structures," Journal of Structural Engineering, vol. 110, No. 10, Oct. 1984.
- [4] Collins, J. A., Failure of Materials in Mechanical Design, John Wiley & Sons, 1981.
- [5] Dowling, N. E., "Deformation, Fracture, and Fatigue of Engineering Materials," Virginia Polytechnical U., 1989. (manuscript in development)

C-3 Modified Goodman Model High Cycle Fatigue Combines Fatigue and Yield Failure Modes (Constant Amplitude Stress)

- Model for Strength



Modified Goodman range-of-stress diagram for fatigue failure in N cycles.

σ_a = alternating stress amplitude

σ_m = mean stress

$$\sigma_{max} = \sigma_m + \sigma_a$$

$$\sigma_{min} = \sigma_m - \sigma_a$$

σ_m : mean stress

S_u : ultimate strength

S_N : fatigue strength at N cycle

S_y : yielding strength

• Failure Prediction Equations :

$$\sigma_{max} - 2\sigma_m \geq S_y, \quad \text{for } -S_y \leq \sigma_m \leq (S_N - S_y)$$

$$\sigma_{max} - \sigma_m \geq S_N, \quad \text{for } (S_N - S_y) \leq \sigma_m \leq 0$$

$$\sigma_{max} - (1 - r)\sigma_m \geq S_N, \quad \text{for } 0 \leq \sigma_m \leq \frac{(S_y - S_N)}{(1 - r)}$$

$$\sigma_{max} \geq S_y, \quad \text{for } \frac{(S_y - S_N)}{(1 - r)} \leq \sigma_m \leq S_y$$

where $r = S_N/S_u$

• Reliability Considerations :

$$P_f = P_r[S_y < \sigma_a - \sigma_m], \quad \text{for } -S_y \leq \sigma_m \leq (S_N - S_y)$$

$$P_f = P_r[S_N < \sigma_a], \quad \text{for } (S_N - S_y) \leq \sigma_m \leq 0$$

$$P_f = P_r[S_N < r\sigma_m + \sigma_a], \quad \text{for } 0 \leq \sigma_m \leq \frac{(S_y - S_N)}{(1 - r)}$$

$$P_f = P_r[S_y < \sigma_m + \sigma_a], \quad \text{for } \frac{(S_y - S_N)}{(1 - r)} \leq \sigma_m \leq S_y$$

Note: For reliability analysis, let all parameters be random variables. Formulate g-function

$$g = S_y - (\sigma_a - \sigma_m)$$

$$g = S_N - \sigma_a$$

$$g = S_N - (r\sigma_m + \sigma_a)$$

$$g = S_y - (\sigma_m + \sigma_a)$$

Numerically, use AMVFO to estimate $P_f \equiv F_g = 0$

- Reference

- [1] Collins, J. A., Failure of Materials in Mechanical Design, John Wiley & Sons, 1981.

C-4 The General $S - N$ (Non-Linear) Representation of Fatigue Strength; Constant and Variable Amplitude Stress

Case C-1 summarized the characteristic $S - N$ curve, linear in log-log space. But sometimes fatigue strength data is non-linear. This is the case for strain-cycled data (see Case L-2). An example is shown in Fig. 1. Possible models are,

$$\log N = A_0 + \sum_{i=1}^k A_i \epsilon^{a_i} \quad (1)$$

$$\log N = A_0 + \sum_{i=1}^k A_i (\log \epsilon)^{a_i} \quad (2)$$

A_i and a_i are empirical constants of material

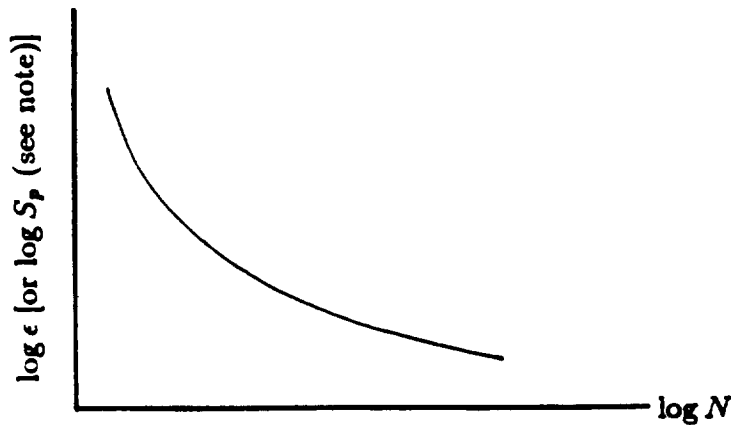


Fig. 1. Non-Linear $S - N$ Relations.

Note: "Pseudo-stress," $S_p = E\epsilon$ is often plotted even though the material is in the plastic range and S_p is not equal to the stress.

For a reliability analysis, it would be necessary to express the parameters as random variables; as a practical matter, the scatter in data might be represented by only one or two random variables.

Other examples of non-linear fatigue strength data are shown in Fig. 2 and Fig. 3. Fig. 2 is a typical description of fatigue behavior given in MIL-HDBK-5B

Fig. 2

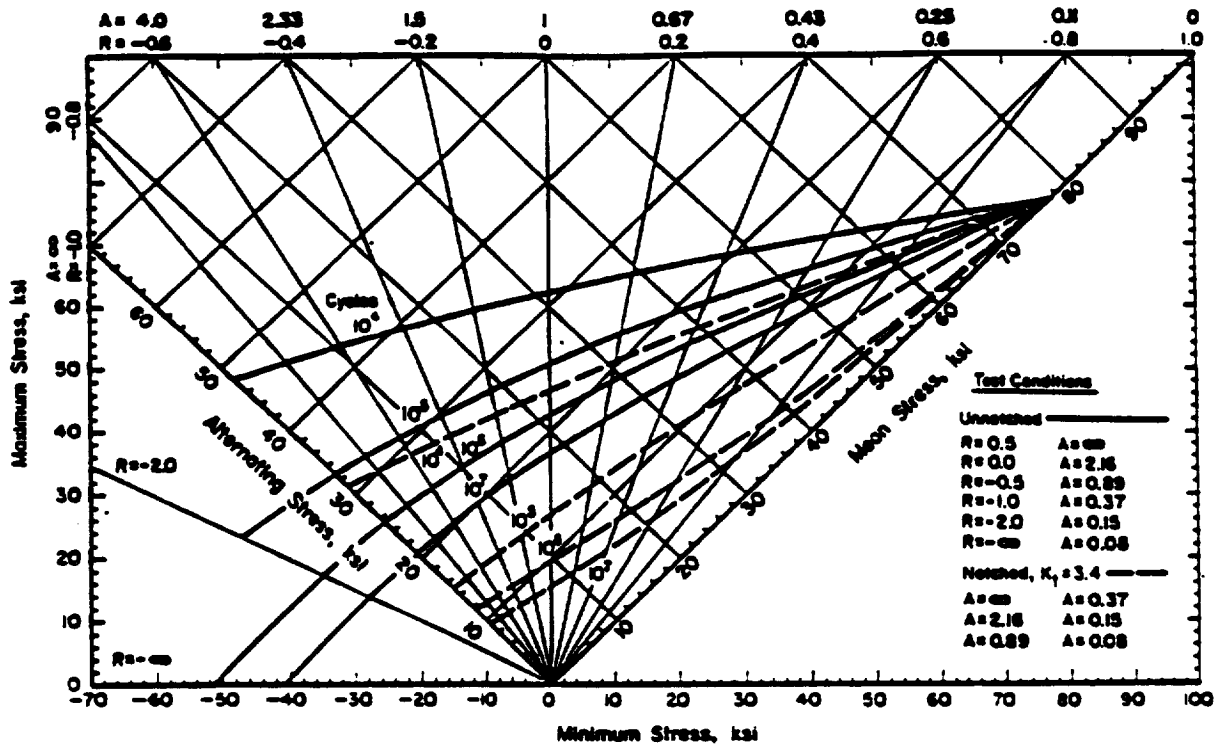


Fig. 2. Typical constant-life diagram for fatigue behavior of various wrought products of 2024-T4 aluminum alloy.

Fig. 3

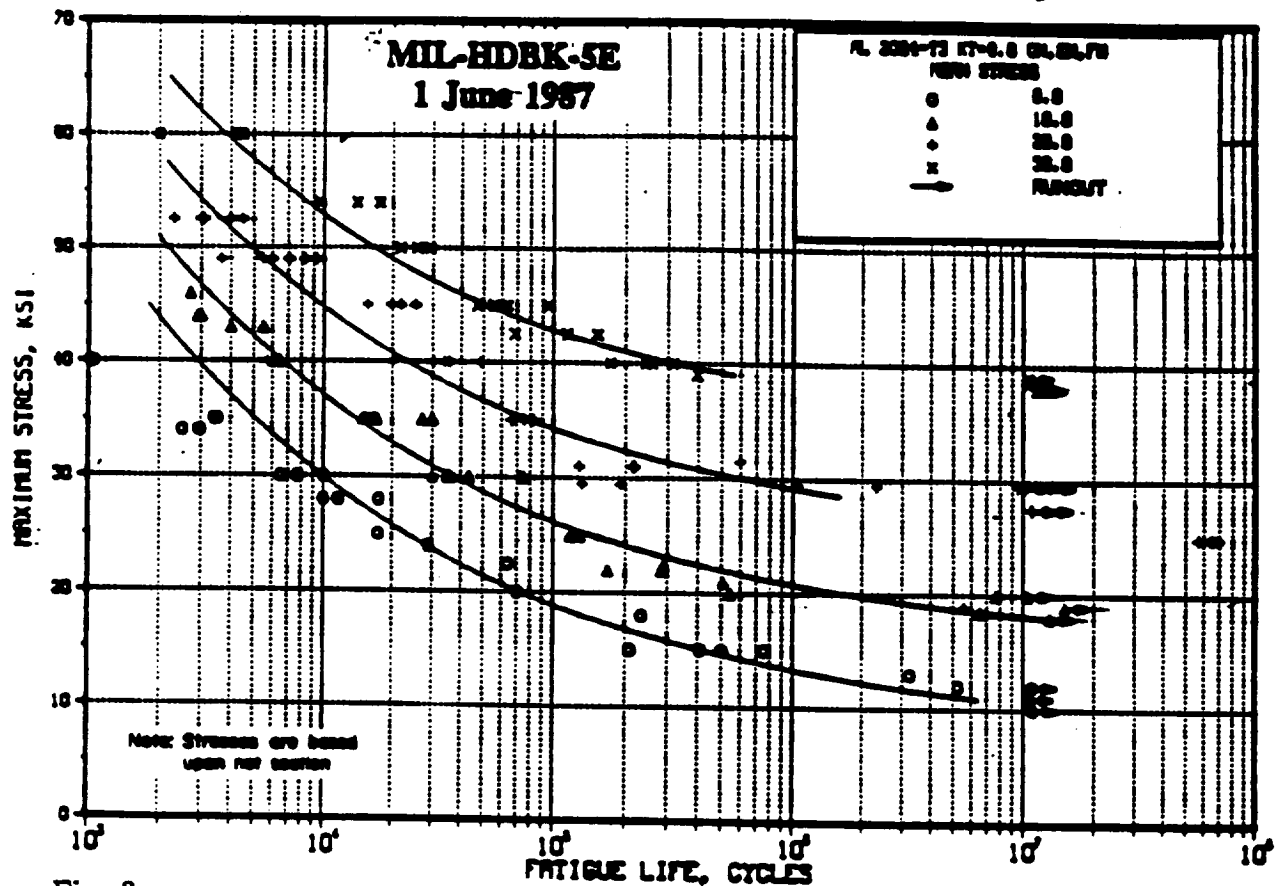


Fig. 3.

Best-fit S/N curves for notched, $K_t = 2.0$, 2024-T3 aluminum alloy sheet longitudinal direction.

Product Form: Bare sheet, 0.090 inch

Test Parameters:

Properties:

TUS, ksi	TYS, ksi	Temp., F
73	54	RT
		(unnotched)
73	—	RT
		(notched $K_t = 2.0$)

Loading - Axial
Frequency - 1100 to 1800 cpm
Temperature - RT
Environment - Air

No. of Heats/Lots: Not specified

Equivalent Stress Equation:

$$\log N_f = 9.2 - 3.33 \log (S_{eq} - 12.3)$$

$$S_{eq} = S_{max} (1-R)^{0.68}$$

Standard Error of Estimate = 0.27

Standard Deviation in Life = 0.89

$R^2 = 91\%$

Sample Size = 113

for various materials. Note that mean stress and stress concentration effects are included. The "newer" representation of fatigue strength is that of MIL-HDBK-5E. The $S - N$ curve is plotted as maximum stress versus fatigue life, with the mean stress specified as illustrated in Fig. 3. Note that in Fig. 3, the non-linear model is of the form,

$$\log N = A_0 - A_1 \log(S - A_2) \quad (3)$$

where the A_i are empirical constants of the material.

- For variable amplitude stress, Miner's damage rule (assuming no sequence effects) is used for the representation. The variable amplitude stress process is represented in block loading as follows:

Block	number of applied cycles	const. amp. stress level
1	n_1	S_1
2	n_2	S_2
3	n_3	S_3
.	.	.
.	.	.
k	n_k	S_k

The accumulated damage is,

$$\begin{aligned} D &= \sum_{i=1}^k D_i \\ &= \sum_{i=1}^k \frac{n_i}{N(S_i)} \end{aligned} \quad (4)$$

where

D : accumulated damage factor

D_i : damage factor due to each block loading

$N(S_i)$: number of cycles to failure for that

: stress level and mean value

- Reliability Considerations:

1. Failure events are:

- a. $[N < N_0]$ for constant amplitude stress process; where N_0 is design life.
- b. $[D > \Delta]$ for variable amplitude stress process; where Δ is damage at failure.

2. Probability of failure:

- a. $P_f = \Pr[N < N_0]$
- b. $P_f = \Pr[D > \Delta]$

Parameters associated with N and D , and N_0 , Δ may all be random variables. The reliability analysis can be made using Wu/FPI or AMVFO.

• References:

- [1] Bannantine, J. A., Comer, J. J., and Handrock, J. L., Fundamentals of Metal Fatigue Analysis, Prentice-Hall, 1990.
- [2] U.S. Department of Defense, MIL-HDBK-5.

C-5 Characteristic S - N Curve Variable Amplitude Stresses

- Fatigue damage

$$D = \frac{n}{A} E[S^m]$$

Parameters :

- Fatigue damage, D
- Total number of cycles, n
- Fatigue strength coefficient, A
- Fatigue strength exponent, m
- Applied stress range, S
- ($E[\cdot]$ denotes the expected value)

- Assumptions:

1. Miner's rule applies. (No stress sequence effects)
2. Basic fatigue curve is $NS^m = A$. (no endurance limit)
3. Stationary and zero mean stress processes.
4. The spectral density function of stress process is given, $W(f)$.
5. Time of application of stress process is T .

- Information obtained from $W(f)$

1. k_{th} moment of $W(f)$, $m_k = \int_0^\infty f^k W(f) df$
2. root-mean-square of the process, $\sigma = \sqrt{m_0}$
3. expected rate of zero crossing with (+) slope, $n_0 = \sqrt{m_2/m_0}$
4. expected rate of peaks, $n_p = \sqrt{m_4/m_2}$
5. irregularity factor, $\alpha = n_0/n_p$
6. spectral width parameter, $\epsilon = \sqrt{1 - \alpha^2}$

- Case A : Stress process is stationary, Gaussian, and narrow band.

Probability density function of stress range S_R

$$f_{S_R}(r) = \frac{r}{4\sigma^2} e^{-\frac{r^2}{8\sigma^2}} \quad \text{for } 0 \leq r < \infty$$

Fatigue damage factor

$$D_{NB} = \frac{n}{A} (2\sqrt{2}\sigma)^m \Gamma\left(\frac{m}{2} + 1\right)$$

in which $\Gamma(\cdot)$ denotes gamma function. Here A is based on stress range. If A is based on amplitude, use $(\sqrt{2}\sigma)^m$ to replace $(2\sqrt{2}\sigma)^m$.

- Case B : Stress process is stationary, Gaussian, and wide band.

1. Equivalent narrow band approximation (Wirsching and Light, 1979)

$$D_{WB} = \lambda(m, \epsilon) D_{eq, NB}$$

where

$$\begin{aligned} D_{eq, NB} &= \frac{\nu_0 T}{A} (2\sqrt{2}\sigma)^m \Gamma\left(\frac{m}{2} + 1\right) \\ \lambda(m, \epsilon) &= a(m) + [1 - a(m)](1 - \epsilon)^{b(m)} \\ a(m) &= 0.926 - 0.033m \\ b(m) &= -2.323 + 1.587m \end{aligned}$$

2. Equivalent narrow band approximation (Ortiz and Chen, 1987)

$$D_{WB} = \lambda_k D_{eq, NB}$$

where

$$\begin{aligned} \lambda_k &= \frac{\beta_k^m}{\beta_2} \\ \beta_k &= \sqrt{\frac{m_2 m_k}{m_0 m_{k+2}}} \\ k &= \frac{2.0}{m^{0.86}} \end{aligned}$$

3. Local range cycle counting (Ortiz, 1985)

$$D_{WB} = \frac{n_p T}{A} (2\sqrt{2}\alpha\sigma)^m \Gamma\left(\frac{m}{2} + 1\right)$$

- Mean stress effect

If the stress process has a non-zero mean, replace A by A^* . Using the Goodman correction

$$A^* = A_0 \left(1 - \frac{S_m}{S_u}\right)^m$$

where A_0 is obtained from zero mean test; S_m is the mean stress; and S_u is the ultimate strength of the material.

• Reliability Consideration :

1. The event of failure is $[D > \Delta]$ where Δ is damage at failure, in general a random variable[4].
2. The probability of failure is

$$P_f = P_r[D \geq \Delta]$$

where $D = D(A, m, S, n, W)$ all of which could be random. Obtain solution by Wu/FPI or AMVFO.

• References :

- [1] Ortiz, K. and Chen, N.K., "Fatigue Damage Prediction for Stationary Wide-band Random Stresses," ICASP 5, Fifth International Conference on Applications of Statistics and Probability in Soil and Structural Engineering, Vancouver, Canada, May 1987.
- [2] Wirsching, P. H. and Light, M. C., "Fatigue Under Wide Band Random Stresses," Journal of the Structural Division, ASCE, Vol. 108, No. ST7, pp. 1593-1607, 1979.
- [3] Ortiz, K., "On the Stochastic Modeling of Fatigue Crack Growth," Stanford Ph. D. Dissertation, 1985.
- [4] Wirsching, P. H., "Fatigue Reliability for Offshore Structures," Journal of Structural Engineering, vol. 110, No. 10, Oct. 1984.

C-6 Characteristic S-N Curve With No Endurance Limit; Variable Amplitude Stresses Including Sequence Effects; General Variable Amplitude Stress; Stationary Gaussian Processes Including Both Narrow Band And Wide Band; Mean Stress.

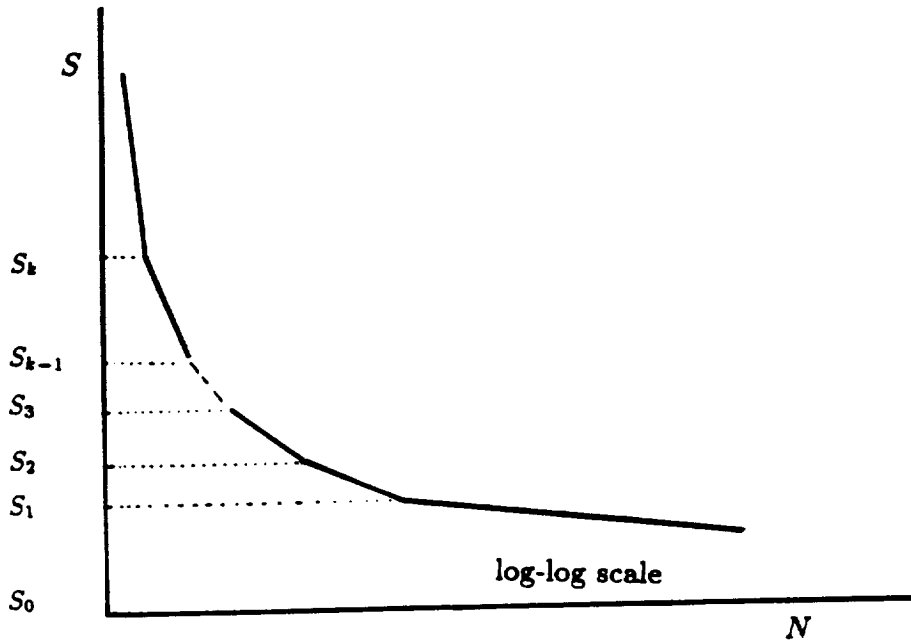
- This case of fatigue models, which accounts for variable amplitude stresses including sequence effects in a characteristic $S - N$ format, is still under review (see also Case F-5). In practice, sequence effects are ignored in life prediction estimates in most application.

C-7 Multi-Linear $S - N$ Curve (Under Variable Amplitude Stress)

- Basic Assumptions:

1. Stress sequence effects are negligible.
2. f_S , the probability distribution function of S , is known.
3. Miner's rule applies.

- Model for fatigue strength



$$NS^{m_1} = A_1, \quad \text{for } S_0 \leq S < S_1$$

$$NS^{m_2} = A_2, \quad \text{for } S_1 \leq S < S_2$$

$$NS^{m_3} = A_3, \quad \text{for } S_2 \leq S < S_3$$

⋮

$$NS^{m_k} = A_k, \quad \text{for } S_{k-1} \leq S < S_k$$

or, in general form,

$$NS^{m_i} = A_i, \quad \text{for } S_{i-1} \leq S < S_i \quad i = 1, 2, 3, \dots, k$$

where

S : applied stress range or amplitude

A_i : fatigue strength coefficient

m_i : fatigue strength exponent

Let $S_0 = 0$ and $S_\infty = \infty$.

- Fatigue Damage, D [1]:

$$\begin{aligned} D &= \frac{n}{A} E[S^m] \\ &= \frac{n}{A} \int_{\Omega} s^m f_S(s) ds \end{aligned}$$

where n is the number of cycles of the applied stress range, $E[\cdot]$ denotes the expected value, Ω is the domain of stress range, and $f_S(s)$ is the probability density function of S .

- For multi-linear $S - N$ curve, fatigue damage, D can be expressed as [2]:

$$\begin{aligned} D &= \frac{n_1}{A_1} E[S^{m_1}] + \frac{n_2}{A_2} E[S^{m_2}] + \dots + \frac{n_k}{A_k} E[S^{m_k}] \\ &= \frac{n_1}{A_1} \int_{S_0}^{S_1} s^{m_1} f_S(s) ds + \frac{n_2}{A_2} \int_{S_1}^{S_2} s^{m_2} f_S(s) ds + \dots + \frac{n_k}{A_k} \int_{S_{k-1}}^{S_k} s^{m_k} f_S(s) ds \\ &= \sum_{i=1}^k \frac{n_i}{A_i} \int_{S_{i-1}}^{S_i} s^{m_i} f_S(s) ds \end{aligned} \tag{1}$$

- Mean Stress Effects:

If $\sigma_m > 0$, then the fatigue strength coefficients, A_i , using Goodman correction, should be

$$A_i = A_{0i} \left[1 - \frac{\sigma_m}{S_u} \right]^{m_i}$$

where A_{0i} is the fatigue strength coefficient for the i th segment for zero mean stress, and S_u is the ultimate strength of the material.

- Reliability Consideration:

1. The failure event is $[D \geq \Delta]$ where Δ is damage at failure. In general, both D and Δ are random variables.

2. The probability of failure is

$$P_f = \Pr[D \geq \Delta]$$

Reliability prediction can be made by Wu/FPI or AMVFO.

• Comments:

1. Existence of endurance limit.

If an endurance limit exists, then the linear section representing the endurance limit is in the form of $NS^{m_1} = A_1$ with $A_1 = \infty$. Thus the first term of Eq. 1. vanishes. Eq. 1. will not lose its generality with the existence of the endurance limit.

2. Stress range is Weibull distributed. The Weibull distribution has the promise of being a generic loading model for a wide variety of structures.

$$f_S(s) = \left(\frac{\xi}{\delta}\right) \left(\frac{s}{\delta}\right)^{\xi-1} \exp\left(-\frac{s}{\delta}\right)^{\xi} \quad 0 < s, \delta, \xi < \infty$$

where ξ = Weibull shape parameter and δ = Weibull scale parameter.

Substituting this pdf into Eq. 1. and making variable transformation, one may obtain the fatigue damage, D :

$$D = \sum_{i=1}^k \frac{n_i}{A_i} \delta^{m_i} \int_{z_{i-1}}^{z_i} t^{a_i-1} e^{-t} dt$$

where

$$z_i = \left(\frac{S_i}{\delta}\right)^{\xi}$$

$$a_i = \frac{m_i}{\xi} + 1$$

Note that the integrand is in the same form as that of Gamma function.

• References

- [1] Wirsching, P. H. and Light, M. C., "Fatigue Under Wide Band Random Stresses," Journal of the Structural Division, ASCE, Vol. 108, No. ST7, pp. 1593-1607, 1979.
- [2] Wirsching, P. H., "Considerations of Probability Based Fatigue Design for Marine Structures," Marine Structure, 1 (1988) 23-45.
- [3] Wirsching, P. H., "Fatigue Reliability for Offshore Structures," Journal of Structural Engineering, vol. 110, No. 10, Oct. 1984.
- [4] Collins, J. A., Failure of Materials in Mechanical Design, John Wiley & Sons, 1981.

L-1 The High Cycle Component of the General Strain-Life Relationship (Constant Amplitude Stress)

- Definition of high cycle fatigue :

There is no sharp transition from low cycle to high cycle fatigue. But typically, cycles to failure greater than 10^4 or 10^5 is considered to be high cycle fatigue.

- General strain-life curve:

The strain amplitude ϵ_a corresponding to N cycles to failure is

$$\begin{aligned}\epsilon_a &= (\epsilon_e)_a + (\epsilon_p)_a \\ &= \frac{\sigma'_f}{E}(2N)^b + \epsilon'_f(2N)^c\end{aligned}\tag{1}$$

where

ϵ_a : strain amplitude, $\epsilon_a = \frac{\Delta\epsilon}{2}$, where $\Delta\epsilon$ = strain range

$(\epsilon_e)_a$: elastic strain amplitude

$(\epsilon_p)_a$: plastic strain amplitude

E : Young's modulus

σ'_f : fatigue strength coefficient

b : fatigue strength exponent

ϵ'_f : fatigue ductility coefficient

c : fatigue ductility exponent

N : Cycles to failure

- In high cycle fatigue, the plastic strain is negligible. Thus

$$\epsilon_a = \frac{\sigma'_f}{E}(2N)^b\tag{2}$$

But

$$\begin{aligned}S_a &= E\epsilon_a \\ &= \sigma'_f(2N_f)^b\end{aligned}\tag{3}$$

Therefore, it is obtained

$$N = \frac{1}{2} \left(\frac{S_a}{\sigma'_f} \right)^{\frac{1}{b}} \quad (4)$$

where S_a is the fatigue strength associated with N cycles.

Note that Eq. 4 is exactly the same form as the basic fatigue relationship (described in C-1)

$$NS^m = A \quad (5)$$

where

$$m = -\frac{1}{b} \quad (6)$$

$$A = \frac{1}{2} (\sigma'_f)^{-\frac{1}{b}} \quad (7)$$

• Reliability Consideration :

The event of failure is

1. $[\sigma_a > S_a]$ where σ_a is the applied stress amplitude. In general, both σ_a and S_a are random variables.
2. $[N_f < N_0]$ where N_0 is the intended service life.

The probability of failure is

1. $P_f = P_r[\sigma_a > S_a]$
2. $P_f = P_r[N < N_0]$

The reliability estimates can be solved by Wu/FPI or AMVFO.

• How to Account for Mean Stresses ?

Let σ_m = applied mean stress

1. Morrow's model [1]:

$$\epsilon_a = \left(\frac{\sigma'_f - \sigma_m}{E} \right) (2N)^b \quad (8)$$

Or, in another form

$$\begin{aligned} S_a &= E\epsilon_a \\ &= (\sigma'_f - \sigma_m)(2N)^b \end{aligned} \quad (9)$$

$$N = \frac{1}{2} \left(\frac{S_a}{\sigma'_f - \sigma_m} \right)^{\frac{1}{b}} \quad (10)$$

2. Manson and Halford model [2,3] :

$$\epsilon_a = \left(\frac{\sigma'_f - k_m \sigma_m}{E} \right) (2N)^b \quad (11)$$

where k_m is the mean stress factor to be determined empirically. Express the above equation in another form,

$$\begin{aligned} S_a &= E \epsilon_a \\ &= (\sigma'_f - k_m \sigma_m) (2N)^b \end{aligned} \quad (12)$$

$$N = \frac{1}{2} \left(\frac{S_a}{\sigma'_f - k_m \sigma_m} \right)^{\frac{1}{b}} \quad (13)$$

The Goodman correction can also be applied to these models. Adjust A , as defined above,

$$A = A_0 \left(1 - \frac{\sigma_m}{S_u} \right)^m \quad (14)$$

where

$$\begin{aligned} A_0 &= \frac{1}{2} (\sigma'_f)^{-\frac{1}{b}} \\ S_u &= \text{ultimate strength of material} \end{aligned}$$

• Reliability Consideration :

The event of failure is

1. $[\sigma_a > S_a]$ where σ_a is the applied stress amplitude. In general, both σ_a and S_a are random variables.
2. $[N < N_0]$ where N_0 is the intended service life.

The probability of failure is

1. $P_f = P_r[\sigma_a > S_a]$
2. $P_f = P_r[N < N_0]$

Obtain reliability estimations using Wu/FPI or AMVFO.

• How to account for stress concentrations (the notch effect)?

Stress concentration factor

$$K_t = \frac{\sigma}{\sigma_{nom}} \quad (15)$$

where

K_t = stress concentration factor

σ = local stress amplitude at notch root (point of stress concentration)

S_{nom} = nominal stress amplitude

$$\begin{aligned}\epsilon_a &= \frac{K_t S_a}{E} \\ &= \frac{\sigma'_f}{E} (2N)^b\end{aligned}\tag{16}$$

It is obtained

$$S_a = \frac{\sigma'_f}{K_t} (2N)^b\tag{17}$$

$$N = \frac{1}{2} \left(\frac{S_a K_t}{\sigma'_f} \right)^{\frac{1}{b}}\tag{18}$$

But, a fatigue notch factor, K_f , is often employed in fatigue analysis. Fatigue notch factor, K_f , is defined as the ratio of unnotched fatigue strength to notched fatigue strength.

$$K_f = \frac{S_{unnotched}}{S_{notched}}\tag{19}$$

1. Relationship of K_f to K_t by Peterson [3]:

$$\begin{aligned}K_f &= 1 + (K_t - 1) \times q_P \\ &= 1 + \frac{K_t - 1}{1 + \frac{\alpha}{r}}\end{aligned}\tag{20}$$

where α is a material constant dependent on strength and ductility, determined from long life test data for notched and unnotched samples, and r is notch root radius.

2. Relationship of K_f to K_t by Neuber [4]:

$$\begin{aligned}K_f &= 1 + (K_t - 1) \times q_N \\ &= 1 + \frac{K_t - 1}{1 + \sqrt{\rho/r}}\end{aligned}\tag{21}$$

where ρ is a material constant and related to the grain size of material.

Thus, the S_a and N then are in the forms of

1. From Peterson :

$$S_a = \left[\frac{\sigma'_f}{1 + \frac{K_f - 1}{q_P}} \right] (2N)^b \quad (22)$$

$$N = \frac{1}{2} \left[\frac{S_a}{\sigma'_f} \left(1 + \frac{K_f - 1}{q_P} \right) \right]^{\frac{1}{b}} \quad (23)$$

2. From Neuber :

$$S_a = \left[\frac{\sigma'_f}{1 + \frac{K_f - 1}{q_N}} \right] (2N)^b \quad (24)$$

$$N = \frac{1}{2} \left[\frac{S_a}{\sigma'_f} \left(1 + \frac{K_f - 1}{q_N} \right) \right]^{\frac{1}{b}} \quad (25)$$

where q_P and q_N are defined above.

• Comments :

1. For ferrous-based wrought metals, α is approximately by [5]:

$$\alpha = \left[\frac{300}{S_u(\text{ksi})} \right]^{1.8} \times 10^{-3} \text{ in.} \quad (26)$$

or using the approximation ($S_u \approx 0.5 \text{ BHN}$)

$$\alpha = \left[\frac{300}{0.5 \text{ BHN}} \right]^{1.8} \times 10^{-3} \text{ in.} \quad (27)$$

where BHN is Brinell Hardness Number. Thus,

$\alpha \approx 0.01 \text{ in.}$ for normalized or annealed steels ($\text{BHN} \approx 170$).

$\alpha \approx 0.001 \text{ in.}$ for highly hardened steels ($\text{BHN} \approx 600$).

$\alpha \approx 0.0025 \text{ in.}$ for quench and tempered steels ($\text{BHN} \approx 360$).

2. Typical values of ρ can be referred in Juvinall [6].

• Reliability Consideration :

The event of failure is

1. $[\sigma_a > S_a]$ where σ_a is the applied stress amplitude. In general, both are random variables.
2. $[N < N_0]$ where N_0 is the designed service life.

The probability of failure is

1. $P_f = P_r[\sigma_a > S_a]$
2. $P_f = P_r[N < N_0]$

Reliability estimate can be solved by Wu/FPI or AMVFO.

• References :

- [1] Morrow, J., *Fatigue Design Handbook*, Advances in Engineering, vol. 4, SAE, Warrendale, Pa., 1968, Sec. 3.2, pp. 21-29.
- [2] Manson, S. S. and Halford, G. R., "Practical Implementation of the Double Linear Damage Rule and Damage Curve Approach for Treating Cumulative Fatigue Damage," *Int. J. Fract.*, Vol. 17, No. 2, 1981, pp. 169-172, R35-R42.
- [3] Peterson, R.E., "Analytical Approach to Stress Concentration Effects in Aircraft Materials," Technical Report 59-507, *U.S. Air Force — WADC Symp. Fatigue Metals*, Dayton, Ohio, 1959.
- [4] Neuber, H., *Theory of Notch Stresses: Principle for Exact Stress Calculations*, Edwards, Ann Arbor, Mich., 1946.
- [5] Society of Automotive Engineers, *Fatigue Design Handbook*, Vol. 4, SAE, Warrendale, Pa., 1968, p.29.
- [6] Juvinall, R. C., *Engineering Considerations of Stress, Strain and Strength*, McGraw-Hill, New York, 1967.
- [7] Society of Automotive Engineers, *Fatigue Design Handbook*, AE-10, SAE, Warrendale, Pa., 1988.

L-2 The General Strain-Life Relationship and Low Cycle Fatigue (Constant Amplitude Stresses)

- Basic Assumptions:

1. Fatigue crack initiation is defined as "fatigue failure."
2. The external loading is constant amplitude.

This method of life prediction is also used by turbine engine manufacturers. Clearly, frequency of loading is an important factor. Generally, as a practical matter, cycle life N is evaluated using a computer code. One code is Program BROSE, developed at the University of Arizona (by Y.-T. Wu).

- The general strain-life relationship defines fatigue strength for a cycle life N .

$$\begin{aligned}\epsilon_a &= (\epsilon_e)_a + (\epsilon_p)_a \\ &= \frac{\sigma'_f}{E}(2N)^b + \epsilon'_f(2N)^c\end{aligned}\tag{1}$$

where

ϵ_a : strain amplitude, $\epsilon_a = \frac{\Delta\epsilon}{2}$, where $\Delta\epsilon$ = strain range.

$(\epsilon_e)_a$: elastic strain amplitude.

$(\epsilon_p)_a$: plastic strain amplitude.

E : Young's modulus

σ'_f : fatigue strength coefficient, to be determined from fatigue test.

b : fatigue strength exponent, to be determined from fatigue test.

ϵ'_f : fatigue ductility coefficient, to be determined from fatigue test.

c : fatigue ductility exponent, to be determined from fatigue test.

N : cycles to failure

(i) Pseudo fatigue strength associated with N cycles, S_a :

$$\begin{aligned} S_a &= E\epsilon_a \\ &= \sigma'_f(2N)^b + E\epsilon'_f(2N)^c \end{aligned}$$

(ii) Cycles to failure for a given strain range:

The expression for strength is difficult to invert. This approximate expression is due to Manson, *et al.*, [1].

$$N = N_T \left[\left(\frac{\Delta\epsilon}{\Delta\epsilon_T} \right)^{z/c} + \left(\frac{\Delta\epsilon}{\Delta\epsilon_T} \right)^{z/b} \right]^{1/z} \quad (2)$$

where

$$N_T \text{ (Transition life)} = 0.5 \left(\frac{E\epsilon'_f}{\sigma'_f} \right)^{\frac{1}{1-c}} \quad (3)$$

$$\Delta\epsilon_T \text{ (Transition strain)} = 2(\epsilon'_f)^{\frac{1}{(b-c)}} \left(\frac{\sigma'_f}{E} \right)^{\frac{c}{(c-b)}} \quad (4)$$

$$z = \exp \left\{ P \left(\ln \frac{\Delta\epsilon}{\Delta\epsilon_T} \right)^2 + Q \left(\ln \frac{\Delta\epsilon}{\Delta\epsilon_T} \right) + \ln \left[-0.889c \left(\frac{c}{b} \right)^{-0.36} \right] \right\} \quad (5)$$

$$P = -0.001277 \left(\frac{c}{b} \right)^2 + 0.03893 \left(\frac{c}{b} \right) - 0.0927 \quad (6)$$

$$Q = 0.004176 \left(\frac{c}{b} \right)^2 - 0.13500 \left(\frac{c}{b} \right) + 0.2309 \quad (7)$$

• Comments:

1. The empirical constants, i.e. σ'_f , b , ϵ'_f , and c are determined by a fatigue test, but because fatigue testing is expensive, these constants can be estimated from monotonic tests and cyclic $\sigma-\epsilon$ curve as follows:

$$\begin{aligned} \sigma'_f &\approx \sigma_f \\ b &\approx -\frac{n'}{1+5n'} \\ \epsilon'_f &\approx \left(\frac{\sigma'_f}{K'} \right)^{\frac{1}{n'}} \\ c &\approx \frac{b}{n'} \end{aligned}$$

where

σ_f : true stress at fracture of monotonic test.

n' : cyclic strain hardening exponent.

K' : cyclic strength coefficient.

2. Some rules of thumb:

$$\epsilon_a = \frac{\Delta\epsilon}{2} = 0.01 \quad \text{at } N = 10^3$$
$$\sigma_a = \frac{S_u}{2} \quad \text{at } N = 5 \times 10^5$$

3. b varies from -0.05 to -0.12 with a typical average of -0.085, while c varies from -0.5 to -0.8 with typical value -0.6.

• Manson's method of universal slope [2]:

This is a more primitive and earlier form of the general strain-life relationship,

$$\epsilon_a = 1.75\left(\frac{S_u}{E}\right)N^{-0.12} + 0.5(\epsilon_f)^{0.6}N^{-0.6}$$

where S_u is the ultimate strength of the material, and ϵ_f is the true strain at fracture of monotonic test.

(i) Fatigue strength associated with N cycles, S_a :

$$S_a = E\epsilon_a$$
$$= 1.75S_uN^{-0.12} + 0.5E(\epsilon_f)^{0.6}N^{-0.6}$$

(ii) Cycles to failure cannot be expressed explicitly, but can be solved numerically.

• Comment:

The universal slopes method may be a first approximation for the fully reversed strain-life curve for unnotched smooth specimens, based on monotonic tensile properties.

• Statistical Considerations of Fatigue Data

The method of analyzing strain-life fatigue data for the purpose of reliability analysis and design is described in Ref. 3 and 4. Basically, the concept is to fix the exponents b and c as constants equal to the

least squares estimators and "lump" the uncertainty into σ'_f and ϵ'_f , which are treated as random variables.

• Reliability Consideration :

The event of failure is :

1. $[\epsilon > \epsilon_a]$ where ϵ is the strain amplitude at the notch. In general, both ϵ and ϵ_a are random variables.
2. $[N < N_0]$ where N_0 is the intended service life.

The probability of failure is

1. $P_f = P_r[\epsilon > \epsilon_a]$
2. $P_f = P_r[N < N_0]$

Numerically, the reliability estimates can be solved by using Wu/FPI or AMVFO [5].

• Calculation of the Strain Amplitude at the Notch. [see Ref. 6 and 7]

1. Use a finite element program capable of computing the stress and strain at a notch under plastic deformation.
2. Neuber's rule [8]: An approximate method built into local strain analysis computer codes for calculating N .

$$\begin{aligned}\sigma\epsilon &= K_t^2 \sigma_n \epsilon_n \\ &= \frac{(K_t \sigma_n)^2}{E}\end{aligned}$$

where

σ : local stress amplitude at notch root.

ϵ : local strain amplitude at notch root.

K_t : stress concentration factor.

σ_n : nominal stress amplitude.

ϵ_n : nominal strain amplitude.

3. Seeger and Heuler's version of Neuber's rule [9]:

$$\sigma\epsilon = K_p^2 \sigma^* \epsilon^*$$

where

$$K_p = \frac{(\sigma_n) \text{ at onset of general yielding}}{(\sigma_n) \text{ at first notch yielding}} \\ = \frac{\sigma_p}{\sigma_y/K_t}$$

in which σ_p is the particular value of σ corresponding to fully plastic behavior for an ideal elastic, perfectly plastic material having the same yield strength, σ_y , as the real (strain hardening) material. Also

$$\sigma^* = \frac{K_t}{K_p} \sigma_n$$

and the point (σ^*, ϵ^*) lies on the cyclic $\sigma - \epsilon$ curve, i.e.,

$$\epsilon^* = \frac{\sigma^*}{E} + \left(\frac{\sigma^*}{K'}\right)^{1/n'}$$

4. Topper's version of Neuber's rule [10]:

$$\sigma\epsilon = \frac{(K_f \sigma_n)^2}{E}$$

in which K_f is the fatigue notch factor.

Applications of Neuber's rule and its versions require the solution which satisfies both rule (or version) and cyclic $\sigma - \epsilon$ curve

$$\epsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{K'}\right)^{1/n'}$$

where

$K' =$ cyclic strength coefficient.

$n' =$ cyclic strain hardening exponent.

With given σ_n , the state of $\sigma - \epsilon$ can be solved numerically; hence, one may estimate cycles to failure using Eq(2).

- Mean Stress Effects:

1. Morrow's equation [12]:

$$\epsilon_a = \frac{\sigma'_f - \sigma_0}{E} (2N)^b + \epsilon'_f (2N)^c \quad (8)$$

2. Manson and Halford's equation [13]:

$$\epsilon_a = \frac{\sigma'_f - k_m \sigma_0}{E} (2N)^b + \epsilon'_f \left(\frac{\sigma'_f - k_m \sigma_0}{\sigma'_f} \right)^{c/b} (2N)^c \quad (9)$$

The inversion formula of this equation is

$$N = N'_T \left[\left(\frac{\Delta \epsilon}{\Delta \epsilon_T} \right)^{z/c} + \left(\frac{\Delta \epsilon}{\Delta \epsilon_T} \right)^{z/b} \right]^{1/z} \quad (10)$$

where

$$N'_T = 0.5 \left[(2N_T)^{-b} - \frac{2k_m \sigma_0}{E \Delta \epsilon_T} \right]^{-1/b} \quad (11)$$

k_m is mean stress factor; and N_T , $\Delta \epsilon_T$, z , P and Q are defined in Eqs. (3) through (7).

• Mean Stress Relaxation:

Mean stress effects are seen predominantly at longer lives. At high strain amplitudes (0.5% to 1% or above) where plastic strains are significant, mean stress relaxation occurs and the mean stress tends toward zero. Mean stress relaxation can occur in materials that are cyclically stable.

• References :

- [1] Manson, S. S., and Muralidharan, U., "A Single Expression Formula for Inverting Strain-Life and Stress-Strain Relationship," NASA CR-165347, May, 1981.
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- [12] Morrow, J., *Fatigue Design Handbook*, Advances in Engineering, Vol. 4, SAE, Warrendale, Pa., 1968, Sec. 3.2, pp. 21-29.
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L-3 Low Cycle Fatigue Under Random Stresses

- **Basic Assumptions:**

1. Linear damage accumulation rule applies. (no sequence effects)
2. Stress information is obtained by
 - (i) actual record.
 - (ii) simulation based on the power spectral density function of stress..
3. Material's cyclic $\sigma - \epsilon$ is given. The fatigue strength, σ'_f , ϵ'_f , b , and c are given.

- **Procedure:**

1. Obtain the strain history by computer analysis, which automates local strain analysis. (e.g. Program BROSE at the University of Arizona.) Example is shown in Fig. 1. Note the development of hysteresis loops in Fig. 2.

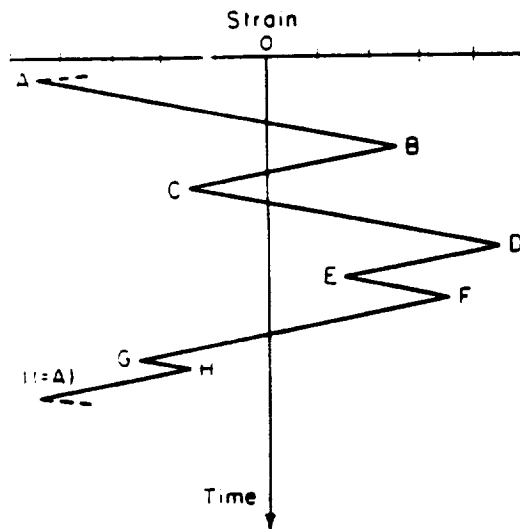


Fig. 1. Example of strain history.

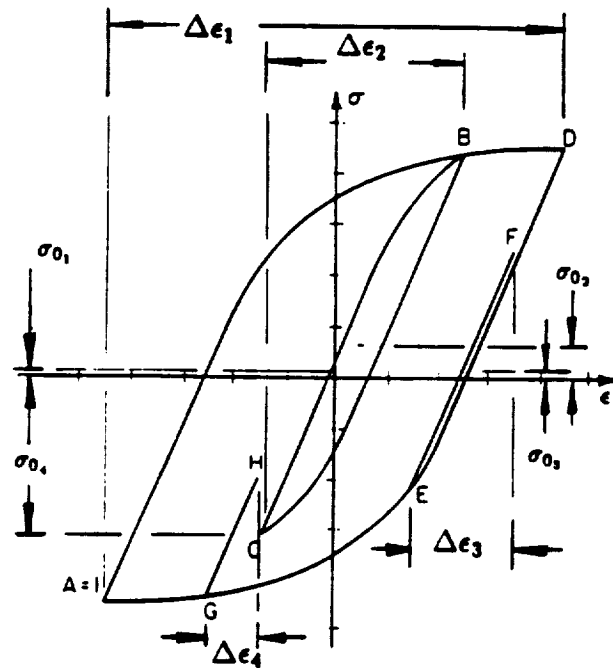


Fig. 2. The hysteresis loops.

2. Using the rainflow cycle counting method, one may determine cycles and the associated mean stresses, σ_0 , and strain ranges, $\Delta\epsilon$ from the hysteresis loops (Fig. 2). In this example, cycles are identified as A-D, B-C, E-F, and

G-H with the strain ranges, $\Delta\epsilon_1$, $\Delta\epsilon_2$, $\Delta\epsilon_3$ and $\Delta\epsilon_4$, respectively. The mean stress are σ_{0_1} , σ_{0_2} , σ_{0_3} , and σ_{0_4} , respectively.

3. For each cycle, i , fatigue life, N_i , can be determined by a strain-life equation that incorporates mean stress effects, such as

i) Morrow's equation [1]:

$$\frac{\Delta\epsilon_i}{2} = \frac{\sigma'_f - \sigma_{0_i}}{E}(2N_i)^b + \epsilon'_f(2N_i)^c \quad (1)$$

ii) Manson and Halford's equation [2]:

$$\frac{\Delta\epsilon_i}{2} = \frac{\sigma'_f - k_m \sigma_{0_i}}{E}(2N_i)^b + \epsilon'_f \left(\frac{\sigma'_f - k_m \sigma_{0_i}}{\sigma'_f} \right)^{c/b} (2N_i)^c \quad (2)$$

where

$\Delta\epsilon_i$: strain range of the i th cycle.

E : Young's modulus.

σ'_f : fatigue strength coefficient.

σ_{0_i} : mean stress of the i th cycle.

b : fatigue strength exponent.

ϵ'_f : fatigue ductility coefficient.

c : fatigue ductility exponent.

k_m : mean stress factor.

N_i : cycles to failure corresponding to the i th load cycle.

4. Total Miner's fatigue damage, D , then is

$$D = \sum_{i=1}^k \frac{1}{N_i} \quad (3)$$

where k is the number of cycles determined from the hysteresis loop.

• Reliability Considerations:

1. The failure event is $[D \geq \Delta]$ where Δ is damage at failure. Δ can be modeled as a random variable.
2. The probability of failure is

$$P_r[D \geq \Delta]$$

Since the fatigue properties $\sigma'_f, \epsilon'_f, \dots$ are random variables, a program to compute D is required. The reliability calculation can be made using AMVFO.

• References :

- [1] Morrow, J., *Fatigue Design Handbook*, Advances in Engineering, Vol. 4, SAE, Warrendale, Pa., 1968, Sec. 3.2, pp. 21-29.
- [2] Manson, S. S. and Halford, G. R., "Practical Implementation of the Double Linear Damage Rule and Damage Curve Approach for Treating Cumulative Fatigue Damage," *Int. J. Fract.*, Vol. 17, No. 2, 1981, pp.169-172, R35-R42.
- [3] Dowling, N. E., Deformation, Fracture, and Fatigue of Engineering Materials, under review for publication as text; VPI, Blacksburg, VA 24060.
- [4] Society of Automotive Engineers, *Fatigue Design Handbook*, AE-10, SAE, Warrendale, Pa., 1988.
- [5] Fuch, H. O., and Stephens, R. I., Metal Fatigue in Engineering, John Wiley & Sons, 1978.

F-1 Fatigue Crack Propagation (Constant Amplitude Stresses)

- Assumptions :

1. No threshold on stress intensity level.
2. Constant amplitude stress. (let S = Stress range)
3. No creep. (high temperature effects)
4. The Paris crack growth law applies.

- Fatigue Crack Growth Law (FCGL)

$$\begin{aligned}\frac{da}{dN} &= C(\Delta K)^m \\ &= C[Y(a)S\sqrt{\pi a}]^m\end{aligned}\tag{1}$$

Parameters :

1. Material empirical constants, C & m .
2. Geometric Correction Factor, $Y(a)$.
3. Applied stress range, S .
4. Initial crack length, a_0 .
5. Failure crack length, a_f , chosen
 - (a) by engineering judgement.
 - (b) critical crack length, i.e., $a_f = K_c^2/(\pi Y^2 S^2)$ where K_c is fracture toughness.

- Case A : Cycles to failure

$$N_f = \frac{1}{C(S\sqrt{\pi})^m} \int_{a_0}^{a_f} \frac{da}{[Y(a)\sqrt{a}]^m}\tag{2}$$

- Case B : Cycles to failure (assume geometry factor is constant)

$$N_f = \frac{2(a_f^{1-\frac{m}{2}} - a_0^{1-\frac{m}{2}})}{(2-m)C(YS\sqrt{\pi})^m}\tag{3}$$

- Case C : Cycles to failure (assume constant geometry factor and $a_0 \ll a_f$)

$$N_f = \frac{2}{a_0^{\frac{m}{2}-1}(m-2)C(YS\sqrt{\pi})^m}\tag{4}$$

• Reliability Considerations :

1. The event of failure is $[N < N_0]$, where N_0 is the design life.
2. The probability of failure is

$$P_f = P_r[N < N_0]$$

where $N = N(a_0, a_f, C, m, Y, S)$, all of which can be random variables. Obtain solution using Wu/FPI or AMVFO.

• Consideration of Mean Stress Effect

Define stress ratio R

$$\begin{aligned} R &= \frac{K_{min}}{K_{max}} \\ &= \frac{Y S_{min} \sqrt{\pi a}}{Y S_{max} \sqrt{\pi a}} \\ &= \frac{S_{min}}{S_{max}} \end{aligned} \quad (5)$$

R is constant since constant amplitude stress.

1. Forman's equation [1] :

$$\frac{da}{dN} = \frac{C \Delta K^m}{(1-R)K_c - \Delta K} \quad (6)$$

- a. Cycles to failure

$$N = \int_{a_0}^{a_f} \frac{(1-R)K_c - \Delta K}{C(\Delta K)^m} da \quad (7)$$

- b. Cycles to failure (assume constant geometry factor)

$$N = \frac{(1-R)K_c(a_0^{1-\frac{m}{2}} - a_f^{1-\frac{m}{2}})}{(\frac{m}{2}-1)C(Y S \sqrt{\pi})^m} - \frac{a_0^{\frac{3-m}{2}} - a_f^{\frac{3-m}{2}}}{(\frac{m-3}{2})C(Y S \sqrt{\pi})^{m-1}} \quad (8)$$

2. Walker's equation [2] :

$$\frac{da}{dN} = C[(1-R)^k K_{max}]^m \quad k \text{ is an empirical constant} \quad (9)$$

a. Cycles to failure

$$N_f = \frac{1}{C[S_{max}(1-R)^k\sqrt{\pi}]^m} \int_{a_0}^{a_f} \frac{da}{[Y(a)\sqrt{a}]^m} \quad (10)$$

b. Cycles to failure (assume constant geometry factor)

$$N = \frac{a_0^{1-\frac{m}{2}} - a_f^{1-\frac{m}{2}}}{(\frac{m}{2} - 1)C(1-R)^{km}(YS_{max}\sqrt{\pi})^m} \quad (11)$$

3. equation suggested by Rolfe and Barsom [3] :

$$\frac{da}{dN} = C[(1-R)^k \Delta K]^m \quad k \text{ is an empirical constant} \quad (12)$$

a. Cycles to failure

$$N_f = \frac{1}{C[S(1-R)^k\sqrt{\pi}]^m} \int_{a_0}^{a_f} \frac{da}{[Y(a)\sqrt{a}]^m} \quad (13)$$

b. Cycles to failure (assume constant geometry factor)

$$N = \frac{a_0^{1-\frac{m}{2}} - a_f^{1-\frac{m}{2}}}{(\frac{m}{2} - 1)C(1-R)^{km}(YS\sqrt{\pi})^m} \quad (14)$$

• Comment:

1. Stress ratio data should be available to fit the empirical constant k

• Reliability Considerations :

1. The event of failure is $[N < N_0]$, where N_0 is the design life.
2. The probability of failure is

$$P_f = P_r[N < N_0]$$

where $N = N(a_0, a_f, C, k, m, Y, S, R, K_c)$, all of which can be random variables. Obtain solution using Wu/FPI or AMVFO.

• References :

- [1] Fuch, H. O., and Stephens, R. I., Metal Fatigue in Engineering, John Wiley & Sons, 1978.
- [2] Walker, K., "The Effect of Stress Ratio during Crack Propagation and Fatigue for 2024-T3 and 7075-T6 Aluminum," ASTM STP 462, 1970, p. 1.
- [3] Rolfe, S. T. and Barsom, J. M., Fracture and Fatigue Control in Structures, Prentice-Hall, 1977
- [4] Broke, D., Elementary Engineering Fracture Mechanics, Martinus Nijhoff Publishers, The Hague, 1984.
- [5] Dowling, N. E., Deformation, Fracture, and Fatigue of Engineering Materials, under review for publication as text; VPI, Blacksburg, VA 24060.

F-2 Fatigue Crack Propagation (Variable Amplitude Stresses)

• Assumptions :

1. No threshold on stress intensity level.
2. The long term statistical distribution of fatigue stresses is known.
3. The Paris crack growth law applies.
4. Sequence effects are negligible.
5. No mean stress effect associated with individual stress cycles.

• Fatigue Crack Growth Law (FCGL)

$$\begin{aligned}\frac{da}{dN} &= C(\Delta K)^m \\ &= C[Y(a)S\sqrt{\pi a}]^m\end{aligned}\quad (1)$$

Parameters :

1. Material empirical constants, Paris coefficient and exponent, C & m .
2. Geometry Correction Factor, $Y(a)$.
3. The distribution parameters of applied stress range, S .
4. Initial crack length, a_0 .

Equation (1) may be integrated in the form as follows [1,2] :

$$\int_0^N S^m dN = \int_{a_0}^{a_N} \frac{da}{C[Y(a)\sqrt{\pi a}]^m} \quad (2)$$

where a_N is crack length after N cycles.

Because of the variable amplitude, the stress cycles are discrete, and equation (2) can be written as,

$$\sum_{i=1}^N S_i^m = \int_{a_0}^{a_N} \frac{da}{C[Y(a)\sqrt{\pi a}]^m} \quad (3)$$

In general, a numerical procedure is required to compute a_N .

• Reliability Considerations :

1. This problem can be thought of as a first passage problem. Failure is assumed to occur the first time that crack length exceeds a_c , a critical crack size

$$a_c \leq a_N \quad (4)$$

The critical crack size is chosen

- a) by engineering judgement.
 - b) by $a_c = K_c^2/(\pi Y^2 S^2)$ where K_c is the fracture toughness of the material.
2. The probability of failure is

$$P_f = \Pr[a_c \leq a_N] \quad (5)$$

where $a_N = a_N(a_0, C, m, Y, S)$, all of which can be random variables. Reliability estimate can be obtained using Wu/FPI or AMVFO.

- In the case where stress is a stationary random process
Rewrite equation (3) as

$$\begin{aligned} \frac{1}{N} \int_{a_0}^{a_N} \frac{dx}{C[Y(a)\sqrt{\pi x}]^m} &= \frac{1}{N} \sum_{i=1}^N S_i^m \\ &= E(S^m) \end{aligned} \quad (6)$$

where $E(\cdot)$ denotes expected value.

For example, let S be Weibull distributed; its probability density function is

$$f_S(s) = \left(\frac{\xi}{\delta}\right) \left(\frac{s}{\delta}\right)^{\xi-1} \exp\left(-\frac{s}{\delta}\right)^{\xi} \quad 0 < s, \delta, \xi < \infty$$

where ξ = Weibull shape parameter and δ = Weibull scale parameter. It can be shown that $E(S^m)$ is,

$$\begin{aligned} E(S^m) &= \int_0^{\infty} s^m f_S(s) ds \\ &= \delta^m \Gamma\left(\frac{m}{\xi} + 1\right) \end{aligned} \quad (7)$$

Define S_0 by the expression,

$$\Pr(S > S_0) = \frac{1}{N} \quad (8)$$

that is, the return period of $(S > S_0)$ is N . It follows from the definition of S_0 that,

$$\begin{aligned} E(S^m) &= S_0^m (\log N)^{-\frac{m}{\xi}} \Gamma\left(\frac{m}{\xi} + 1\right) \\ &= [S_0 (\log N)^{-\frac{1}{\xi}} \Gamma^{1/m}\left(\frac{m}{\xi} + 1\right)]^m \\ &= (S_{eq})^m \end{aligned} \quad (9)$$

S_{eq} is called the equivalent stress range.

Equation (4) is now in the form of

$$N S_{eq}^m = \int_{a_0}^{a_N} \frac{dx}{C[Y(a)\sqrt{\pi x}]^m} \quad (10)$$

Note that this form is identical to the constant amplitude form. This is the fracture mechanics equivalent of Miner's rule.

• References :

- [1] Madsen, H. O., Skjong, R., and Kirkemo, F., "Probabilistic Fatigue Analysis of Offshore Structures — Reliability Updating Through Inspection Results," A.S. Veritas, Hovik, Norway.
- [2] Perng, Horng-Linn, "Damage Accumulation in Random Loads," Ph. D. Dissertation, The University of Arizona, 1989.
- [3] Wirsching, P. H., "Considerations of Probability Based Fatigue Design for Marine Structures," Marine Structure, 1 (1988) 23-45.
- [4] Fatigue Handbook, Offshore Steel Structures, Ed., A. Almar-Wess, Tapir Publisher, 1985.
- [5] Collins, J. A., Failure of Materials in Mechanical Design, John Wiley & Sons, 1981.
- [6] Wirsching, P. H., "Fatigue Reliability for Offshore Structures," Journal of Structural Engineering, Vol. 110, No. 10, Oct. 1984.

F-3 Fatigue Crack Propagation (Variable Amplitude Stress with a Threshold Stress Intensity Factor)

• Assumptions :

1. Threshold stress intensity factor ΔK_{th} exists.
2. The Paris law applies in region II and extends through region III.
3. Sequence effects are negligible.
4. The long term statistical distribution of fatigue stress is known.
5. Mean stress effects are ignored, although they could be introduced through Paris coefficient, C , and ΔK_{th} .

• Fatigue Crack Growth Law (FCGL)

$$\begin{aligned}\frac{da}{dN} &= C(\Delta K)^m \\ &= C[Y(a)S\sqrt{\pi a}]^m\end{aligned}\tag{1}$$

Parameters :

1. Material empirical constants, Paris coefficient, C , and Paris exponent, m .
2. Geometry Correction Factor, $Y(a)$.
3. The distribution parameters of applied stress range, S .
4. Initial crack length, a_0 .

• Consideration of the threshold stress intensity factor [1,2]:

ΔK_{th} is defined as a stress intensity value below which the crack will not grow. Thus, a threshold stress range level is

$$S_0(a) = \frac{\Delta K_{th}}{Y(a)\sqrt{\pi a}}\tag{2}$$

Crack growth occurs for a stress cycle only if $S > S_0(a)$. The truncated density function of damaging stress cycles then is

$$h_S(s, a) = \begin{cases} f_S(s)/Q(a) & \text{for } S_0 < s < \infty \\ 0 & \text{otherwise.} \end{cases}\tag{3}$$

where

$$\begin{aligned} Q(a) &= \Pr[S > S_0(a)] \\ &= \int_{S_0(a)}^{\infty} f_S(s) ds \end{aligned} \quad (4)$$

and $f_S(s)$ is the probability density function of the applied stress range, S . Consider the Paris fatigue crack growth law using the characteristic stress approach,

$$\frac{da}{dN_d} = C[Y(a)S_{de}(a)\sqrt{\pi a}]^m \quad (5)$$

where

N_d = number of damaging cycles only

S_{ed} = equivalent stress for damaging cycles only.

$$\begin{aligned} [S_{de}(a)]^m &= E(S^m) \\ &= \int_{s_0(a)}^{\infty} s^m h_S(s) ds \\ &= \frac{1}{Q(a)} \int_{s_0(a)}^{\infty} s^m f_S(s) ds \\ &= [\bar{S}_0(a)]^m / Q(a) \end{aligned} \quad (6)$$

Because [2]

$$\frac{da}{dN_d} = \frac{1}{Q(a)} \frac{da}{dN} \quad (7)$$

Then, it follows that,

$$\frac{da}{dN} = C[Y(a)\bar{S}_0(a)\sqrt{\pi a}]^m \quad (8)$$

Define the threshold factor as,

$$G(a) = \frac{[\bar{S}_0(a)]^m}{[S_{eq}]^m} \quad (9)$$

where S_{eq} is the Miner's equivalent stress range derived in section F-2.

It follows from equation (9) of section F-2

$$N S_{eq}^m = \frac{1}{C} \int_{a_0}^{a_N} \frac{da}{G(a)[Y(a)\sqrt{\pi a}]^m} \quad (10)$$

In general, a numerical procedure is required to compute a_N , crack size after life N . A computer program has been developed at the University of Arizona to compute N as a function of a_N and a_N as a function of N .

• Reliability Considerations :

1. This problem can be thought of as a first passage problem. Failure is assumed to occur the first time that crack length exceeds a_c , a critical crack size

$$\text{Failure} = [a_c \leq a_N]$$

The critical crack size is chosen

- (a) by engineering judgement.
 - (b) by $a_c = K_c^2/(\pi Y^2 S^2)$, where K_c is the fracture toughness of the material.
2. The probability of failure is

$$P_f = \Pr[a_c \leq a_N] \quad (11)$$

where $a_N = a_N(a_0, C, m, Y, S)$, all of which can be random variables. Reliability estimate can be obtained using Wu/FPI or AMVFO.

• References :

- [1] Wirsching, P. H., "Considerations of Probability Based Fatigue Design for Marine Structures," Marine Structure, 1 (1988) 23-45.
- [2] Wirsching, P. H., "Fatigue Reliability for Offshore Structures," Journal of Structural Engineering, Vol. 110, No. 10, Oct. 1984.
- [3] Fatigue Handbook, Offshore Steel Structures, Ed., A. Almar-Wess, Tapir Publisher, 1985.
- [4] Perng, Horng-Linn, "Damage Accumulation in Random Loads," Ph. D. Dissertation, The University of Arizona, 1989.
- [5] Madsen, H. O., Skjong, R., and Kirkemo, F., "Probabilistic Fatigue Analysis of Offshore Structures — Reliability Updating Through Inspection Results," A.S. Veritas, Hovik, Norway.
- [6] Collins, J. A., Failure of Materials in Mechanical Design, Wiley, 1981.

F-4 Other Models to Describe Fatigue Crack Propagation

- Paris law is a popular model and is presented in sections F-1, F2, and F3. Other empirical models that have been proposed to describe fatigue crack propagation are listed as follows.

$$\frac{da}{dN} = C_1 \overline{\Delta K}^{C_2} \quad (1)$$

$$\frac{da}{dN} = P_1 \frac{(\Delta K - \Delta K_t)^{P_2}}{(\Delta K_c - \Delta K)^{P_3}} \quad (2)$$

$$\frac{1}{da/dN} = \frac{A_1}{(\Delta K)^{n_1}} + A_2 \left[\frac{1}{(\Delta K)^{n_2}} - C' \right] \quad (3)$$

$$\frac{da}{dN} = C(K_{max})^m [(K_{max} + K_e)(1 - R_{eff}) + *K]^2 \quad (4)$$

$$\log_{10} \left(\frac{da}{dN} \right) = P_1 \exp(P_2 \log \Delta K) + P_3 \exp(P_4 \log \Delta K) + P_5 \quad (5)$$

$$\log_{10} \left(\frac{da}{dN} \right) = C_1 \sinh[C_2(\log \Delta K + C_3)] + C_4 \quad (6)$$

$$\frac{da}{dN} = e + (\nu - e) \left[-\ln \left(1 - \frac{\Delta k}{K_b} \right) \right]^{1/k} \quad (7)$$

$$\frac{da}{dN} = \exp[\beta_0 + \beta_1(\log \Delta K) + \beta_2(\log \Delta K)^2 + \beta_3(\log \Delta K)^3] \quad (8)$$

- Comments on these models is provided in the following :

1. Equation (1) is a modified Paris Law in which

$$\overline{\Delta K} = \frac{\Delta K_{eff} - \Delta K}{1 - (\Delta K_{max}/K_c)^2} \quad (9)$$

and

$$K_{eff} = (1 - P_0/P_{max})K_{max} \quad (10)$$

The ratio P_0/P_{max} is calculated from a crack closure model under plane-strain conditions.

2. In equation (2), ΔK_t and ΔK_c are constants defining the asymptotes at low and high ΔK , respectively. P_1 shifts the (log-log) curve up or down, and P_2 and P_3 influence curvature.
3. Equation (3) is another form of the three-component model first proposed by Hudak, *et al.*, [1] for a constant-load ratio

$$\frac{1}{da/dN} = \frac{A_1}{(\Delta K)^{n_1}} + \frac{A_2}{(\Delta K)^{n_2}} - \frac{A_2}{[K_c(1-R)]^{n_2}} \quad (11)$$

where A_1, A_2, n_1, n_2 , and K_c are fitting constants, and R is the stress ratio.

4. Equation (5) is similar to equation (6), except that different curvatures in the upper and lower regions are allowed through the choice of the pairs P_1, P_2 and P_3, P_4 . The coefficient shifts the entire curve up or down. These coefficients are determined through curve fitting to the fatigue crack growth rate data.
5. Equation (7) was the four-parameter Weibull equation, which is actually a three-parameter Weibull cumulative distribution function with a fourth parameter added to normalize the stress-intensity variable. The details are referred to [3].
6. Equation (8) is suggested by Kung [5] to describe the crack growth behavior. If let $Y = \log(da/dN)$, $X = \log \Delta K$, then the equation is a cubic polynomial equation. The advantage of this model is the exclusion of nonlinear terms proposed in the other models.

• Reliability Consideration

1. In general, a special numerical procedure is required to compute either crack length a_N after a specified life, say service life, N_s , or the life N when a critical crack length, a_c is reached. The critical crack length a_c is normally chosen
 - a) by engineering decision
 - b) by $a_c = K_c^2/(\pi Y^2 S^2)$ In the former case, failure occurs when a_N exceeds a_c , while in the latter case, $N < N_s$.

2. Thus, the probability of failure is

$$P_f = \Pr[a_c \leq a_N] \quad (12)$$

or

$$P_f = \Pr[N \leq N_s] \quad (13)$$

all of which can be random variables. Reliability analysis can be done using Wu/FPI or AMVFO.

• References :

- [1] Hudak, S. J., Jr., Saxena, A., Bucci, R. J. and Malcom, R. C., "Development of Standard Methods of Testing and Analyzing Fatigue Crack Growth Rate Data," Technical Report AFML-TR-78-40, Air Force Material Laboratory, Ohio, May, 1978.
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- [3] Miller, M. S. and Gallagher, J. P., "An Analysis of Several Fatigue Crack Growth Rate (FCGR) Descriptions," *Fatigue Crack Growth Measurement and Data Analysis, ASTM STP 798*, S. J. Hudak, Jr. and R. J. Bucci, Eds., American Society for Testing and Materials, 1981, pp. 205-251.
- [4] Yang, J. N., Salivar, G. C. and Annis, C. G., Jr., "Statistical Modeling of Fatigue-Crack Growth In A Nickel-Base Superalloy," *Engineering Fracture Mechanics*, Vol. 18, No. 2, pp. 257-270, 1983.
- [5] Kung, C. J., "Objective Comparisons of Various Fatigue Crack Growth Laws Based On Time Series Analysis Criteria," Master Report, The University of Arizona, 1988.

**F-5 Crack Growth Model for
Variable Amplitude Stresses
(Sequence Effects are Significant)**

• **Crack-tip plasticity models :**

Assumptions :

1. Load interaction effects (crack growth retardation) occur due to the large plastic zone developed during the overload.
2. Load interaction effects remain active as long as the crack-tip plastic zone developed on the following cycles remains within the plastic zone of the overload.

a. Wheeler model[1] :

$$a_n = a_0 + \sum_{i=1}^n (C_p)_i f(\Delta K)_i$$

Parameters

a_n = crack length after n cycles of load application.

a_0 = initial crack length.

$f(\Delta K)$ = the fatigue crack growth law, e.g., Paris law.

C_p = an empirical retardation parameter.

How is C_p defined ?

$$(C_p)_i = \left[\frac{r_{yi}}{(a_{OL} + r_{OL}) - a_i} \right]^p \quad \text{if } a_i + r_{yi} < a_{OL} + r_{OL}$$

$$(C_p)_i = 1 \quad \text{if } a_i + r_{yi} \geq a_{OL} + r_{OL}$$

where

r_{yi} = the plastic zone size due to the i th loading cycle.

$$r_{yi} = \frac{1}{\beta\pi} \left(\frac{K_{maxi}}{S_y} \right)^2$$

$\beta = 2$ for plane stress case

$\beta = 6$ for plane strain case

a_{OL} = the crack length at a high tensile load application.

r_{OL} = the plastic zone size caused by the high load.

$$r_{OL} = \frac{1}{\beta\pi} \left(\frac{K_{OL}}{S_y} \right)^2$$

$$\beta = 2 \quad \text{for plane stress case}$$

$$\beta = 6 \quad \text{for plane strain case}$$

a_i = crack length at i th loading cycle.

p = an empirical shaping exponent to be determined experimentally.

• Comments:

1. A major disadvantage of this model is the empirical shaping exponent, p .
2. This model neglects the counteracting effect of a negative peak load in crack retardation.

b. Willenborg model[2] :

The procedure of this model is outlined below

1. Determine a_p :

$$\begin{aligned} a_p &= a_0 + r_{OL} \\ &= a_0 + \frac{1}{\beta\pi} \left(\frac{K_{OL}}{S_y} \right)^2 \end{aligned}$$

2. Determine the required stress, $(\sigma_{req})_i$:

$$\begin{aligned} a_p &= a_i + \frac{1}{\beta\pi} \left[\frac{(K_{req})_i}{S_y} \right]^2 \\ &= a_i + \frac{1}{\beta\pi} \left[\frac{Y(a)(\sigma_{req})_i \sqrt{\pi a_i}}{S_y} \right]^2 \end{aligned}$$

$$(\sigma_{req})_i = \frac{S_y}{Y(a)} \sqrt{\frac{\beta(a_p - a_i)}{a_i}}$$

3. Determine the compressive self-stress, $(\sigma_{comp})_i$:

$$(\sigma_{comp})_i = (\sigma_{req})_i - (\sigma_{max})_i$$

where $(\sigma_{max})_i$ is the maximum stress occurring at the i th cycle.

4. Determine an “effective” stress range, $\Delta\sigma_i^e$:

$$\begin{aligned}(\sigma_{max}^e)_i &= (\sigma_{max})_i - (\sigma_{comp})_i \\ &= 2(\sigma_{max})_i - (\sigma_{req})_i\end{aligned}$$

$$\begin{aligned}(\sigma_{min}^e)_i &= (\sigma_{min})_i - (\sigma_{comp})_i \\ &= (\sigma_{max})_i + (\sigma_{min})_i - (\sigma_{req})_i\end{aligned}$$

If either of these “effective” stresses is less than zero, it is set equal to zero.

$$\Delta\sigma_i^e = (\sigma_{max}^e)_i - (\sigma_{min}^e)_i$$

5. Determine $(\Delta K_{eff})_i$ [and $(R_{eff})_i$ if necessary]:

$$(\Delta K_{eff})_i = Y(a) \Delta\sigma_i^e \sqrt{\pi a_i}$$

$$(R_{eff})_i = \frac{(K_{min})_i^{eff}}{(K_{max})_i^{eff}}$$

6. Determine the crack length after n cycles of loading, a_n :

$$a_n = a_0 + \sum_{i=1}^n f(\Delta K_{eff})_i$$

where $f(\cdot)$ denotes the fatigue crack growth law, e.g., Paris law.

• Comments :

1. a_p is the sum of the initial crack length (the crack length when the overload was applied) and the plastic zone due to the overload. Crack growth retardation will decrease until the sum of current crack length, a_i , and its associated plastic zone, r_{yi} , is equal to or large than a_p . In other words, when the boundary of the current plastic zone touches the boundary of the overload plastic zone, retardation ceases.
2. The required stress, $(\sigma_{req})_i$ is the stress required to produce a yield zone, $(r_{req})_i$, whose boundary just touches the overload plastic zone boundary.
3. The most significant difference between the Wheeler and Willenborg models is that the Willenborg model uses only constant-amplitude crack growth data and does not require a “shaping” exponent.

• **Crack Closure Model[3] :**

Assume crack extension occurs only when the applied stress is greater than the crack opening stress, S_{op} .

$$\frac{da}{dN} = f(\Delta K_{eff})$$

where

$f(\cdot)$ = fatigue crack growth law, e.g., Paris law

$$\begin{aligned}\Delta K_{eff} &= K_{max} - K_{op} \\ &= (S_{max} - S_{op})\sqrt{\pi a}Y(a) \\ &= S_{max}(1 - CF)\sqrt{\pi a}Y(a)\end{aligned}$$

CF = Correction Factor

$$= \frac{S_{op}}{S_{max}}$$

The procedure of this model is as follows:

1. Determine $(\Delta S_{eff})_i$ and consequently ΔK_{eff}

$$(\Delta S_{eff})_i = (S_{max})_i - (S_{op})_i$$

2. Calculate Δa_i

$$\begin{aligned}\Delta a_i &= \left(\frac{da}{dN}\right)_i \\ &= f(\Delta K_{eff})_i\end{aligned}$$

3. Determine a_{i+1} :

$$a_{i+1} = a_i + \Delta a_i$$

Repeat these steps until final crack length, say, a_n , exceeds some critical crack length, say, a_f .

• **Comments:**

1. Large computer programs with long run times are often required.
2. Good correlations have been obtained between predicted and experimental results.
3. When Paris law is applied, the coefficient C must correspond to the same closure level as the effective stress intensity factor term, ΔK_{eff} . That is,

$$\left(\frac{da}{dN}\right) = C(\Delta K_{eff})^m$$

$$C = \frac{C^*}{U^m}$$

where C^* = crack growth coefficient obtained from constant amplitude stress range test.

$$U = \frac{\Delta K_{eff}}{\Delta K}$$

$$\Delta K_{eff} = K_{max} - K_{op}$$

$$\Delta K = K_{max} - K_{min}$$

The crack growth exponent, m , does not need to be modified to account for crack closure effects.

• Reliability Consideration :

1. The event of failure is $[a_n > a_f]$, where a_f is specified failure crack length which might be a random variable.
2. The probability of failure is

$$P_f = P_r[a_n > a_f]$$

Reliability estimate can be obtained using Wu/FPI or AMVFO.

• Computer Programs for Computing Life using Cycle-By-Cycle Counting :

1. NASA/FLAGRO, NASA Johnson
2. ASDGRO, Aeronautical Systems Division, WPAFB, OH
3. The Fracture Mechanic, American Society of Metals, 1986
4. Fast-2, Newman, J. C., NASA Langley.
5. CRACKS II. [5]
6. EFFGRO [6]

• References :

- [1] Wheeler, O. E., "Spectrum Loading and Crack Growth," J. Basic Eng., Trans. ASME, Vol. D94, No. 1, 1972, pp. 181-186.
- [2] Willenborg, J., Engle, R. M., and Wood, H. A., "A Crack Growth Retardation Model Using An Effective Stress Concept," AFFDL TM-71-1-FBR, Jan. 1971.
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- [4] Fuch, H. O., and Stephens, R. I., Metal Fatigue in Engineering, John Wiley & Sons, 1978.
- [5] Engle, R. M., "CRACKS II User Manual," AFFDL TM-173-FBE, Aug. 1974.
- [6] Chang, J. B., "Improved Methods for Predicting Spectrum Loading Effects," Rockwell International, Los Angeles Division, First Quarterly Report, NA078-491, May 1978.

H-1 High Temperature Low Cycle Fatigue (Synergistic Effects of Creep and Fatigue) Linear and Elliptic Prediction Rules; Constant Amplitude Stress

- Assumptions :

1. Creep behavior is controlled by the creep stress, σ_m .
2. Fatigue behavior is controlled by the stress amplitude, σ_a .
3. Isothermal conditions.

- Life Prediction Rules (Fig. 1):

Notation:

- σ_a : stress amplitude
 σ_N : fatigue strength at N cycles
 σ_m : creep stress
 σ_{cr} : creep-limited static stress

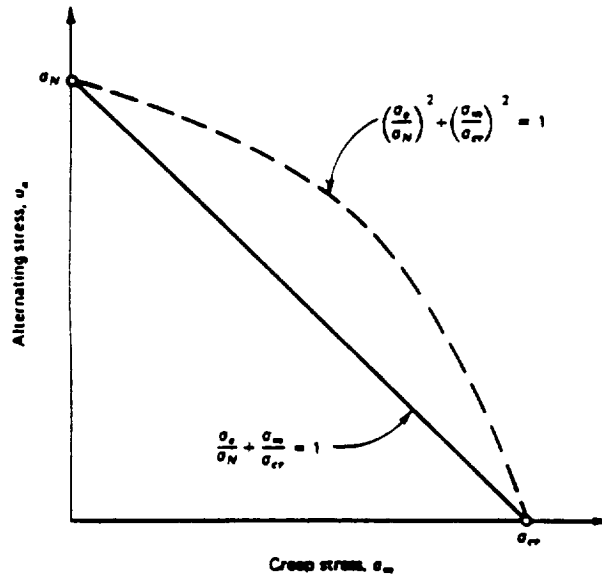


Fig. 1. Failure prediction diagram for combined creep and fatigue under constant temperature conditions.

1. Linear prediction rule:

Failure is predicted to occur under combined isothermal creep and fatigue if

$$D = \frac{\sigma_a}{\sigma_N} + \frac{\sigma_m}{\sigma_{cr}} \geq 1 \quad (1)$$

2. Elliptic prediction rule :

Failure is predicted to occur under combined isothermal creep and fatigue if

$$D = \left(\frac{\sigma_a}{\sigma_N}\right)^2 + \left(\frac{\sigma_m}{\sigma_{cr}}\right)^2 \geq 1 \quad (2)$$

• Comments:

1. This approach is similar to the Goodman linear model and the elliptic model, which accounts for the mean stress depicted in Section C-1.
2. The creep-limited static stress corresponds either to the design limit on creep strain at the design life or to creep rupture at the design life, depending on which mode governs.
3. The linear rule is usually (but not always) conservative. In the higher temperature portion of the creep range, the elliptic relation usually gives better agreement with data.

• Reliability Considerations:

1. The event of failure is $[D \geq 1]$, where D is defined as above.
2. The probability of failure is

$$P_f = \Pr[D \geq 1]$$

All of which can be random variables. Reliability analysis can be made using W_u/FPI or $AMVFO$.

• References :

- [1] Collins, J. A., Failure of Materials in Mechanical Design, John Wiley & Sons, 1981.

**H-2 High Temperature Low Cycle Fatigue
(Synergistic Effects of Creep and Fatigue)
Frequency Modified Strain; Constant Amplitude Stress**

• **Total-Strain-Range-versus-Life Relation [1]:**

The plastic strain-life relationship is

$$\Delta\epsilon_p = \epsilon'_f N^c \nu^{-(1-\kappa)c} \quad (1)$$

The elastic strain-life relationship is

$$\begin{aligned} \Delta\epsilon_e &= \frac{\Delta\sigma}{E} \\ &= \frac{K'}{E} \Delta\epsilon_p^{n'} \nu^\gamma \\ &= \frac{K'}{E} (\epsilon'_f)^{n'} N^{cn'} \nu^{\gamma-(1-\kappa)cn'} \end{aligned} \quad (2)$$

and the total strain-life relationship is

$$\begin{aligned} \Delta\epsilon_t &= \Delta\epsilon_e + \Delta\epsilon_p \\ &= \frac{K'}{E} (\epsilon'_f)^{n'} N^{cn'} \nu^{\gamma-(1-\kappa)cn'} + \epsilon'_f N^c \nu^{-(1-\kappa)c} \end{aligned} \quad (3)$$

where

$\Delta\epsilon_e$: Elastic strain range

$\Delta\epsilon_p$: Plastic strain range

$\Delta\epsilon_t$: Total strain range

K' : Cyclic stress-strain coefficient

E : Young's modulus of the material

ϵ'_f : Fatigue ductility coefficient

n' : Cyclic strain-hardening exponent

N : Applied service life

c : fatigue ductility exponent

ν : Applied frequency

κ, γ : Material's constants

The quantities κ and γ measure the effect of frequency on fatigue life. Coefficients and exponents in equations (1), (2) and (3) can be determined from the results of several laboratory tests at specific strain ranges and frequencies, by means of regression analysis. The coefficient K' and exponent n' can be determined from cyclic stress-strain experiments at unit frequency while γ is obtained in similar experiments by varying the frequency while maintaining the plastic strain-range constant.

From equation (2), one may obtain

$$\Delta\sigma = K'(\epsilon'_f)^{n'} N^{cn'} \nu^{\gamma-(1-\kappa)cn'} \quad (4)$$

and rewrite it as

$$\frac{\Delta\sigma}{2} = \alpha N^\beta \nu^\delta \quad (5)$$

By setting $\nu = 1$, equation (5) is similar to the Basquin equation. The α and β can be determined using the stress-life method, while δ is determined in the same way as the determination of γ as described previously. With the α , β and δ obtained and K' , n' and γ solved previously, one may have ϵ'_f , c and κ solved.

From equation (3), one may calculate the total strain range $\Delta\epsilon_t$ with a given applied stress cycle. However, a numerical computation is required to compute the life N with a given total strain range $\Delta\epsilon_t$.

• Reliability Considerations:

The event of failure is

1. $[\Delta\epsilon > \Delta\epsilon_t]$, where $\Delta\epsilon$ is the applied strain range after N cycles.
2. $[N < N_0]$, where N_0 is the design life.

The probability of failure is

1. $P_f = \Pr[\Delta\epsilon > \Delta\epsilon_t]$
2. $P_f = \Pr[N < N_0]$

All the parameters, i.e., coefficients, exponents, and material constants associated with life prediction may be random variables. Life prediction can be made using Wu/FPI or AMVFO.

• References :

- [1] Coffin, L. F., Jr., "The Effect of Frequency on the Cyclic Strain and Low Cycle Fatigue Behavior of Cast Udimet 500 at Elevated Temperature," *Metallurgical Transactions*, 12(November, 1971):3105-3113.
- [2] Collins, J. A., Failure of Materials in Mechanical Design, John Wiley & Sons, 1981.

**H-3 High Temperature Low Cycle Fatigue
(Synergistic Effects of Creep and Fatigue)
Strain Range Partitioning; Constant Amplitude Stress**

• **Nomenclature**

- B* Intercept of elastic-strain-range-versus-life relations
- b* Exponent on cyclic life for elastic-strain-range-versus-life relations
- C* Intercept of inelastic-strain-range-versus-life relations
- C'* Intercept of equivalent inelastic line for combined creep-fatigue cycles
- c* Exponent on cyclic life for inelastic-strain-range-versus-life relations
- F* Strain fraction
- K* Cyclic strain-hardening coefficient
- N* Applied cycles on zero mean process
- n* Cyclic strain-hardening exponent
- Δ Range of variable
- ϵ strain
- σ stress

subscripts

- c* Compression
- cc* Creep strain in tension, creep strain in compression
- cp* Creep strain in tension, plastic strain in compression
- el* Elastic
- ij* *pp*, *cp*, *pc*, *cc*
- in* inelastic
- pc* plastic strain in tension, creep in compression
- pp* plastic strain in tension, plastic strain in compression

• **Assumptions [1]:**

1. Cyclic life is a function of the inelastic strain range and the type and relative amounts of time-independent strain (plasticity) and time-dependent strain (creep) present in a cyclic stress-strain hysteresis loop.
2. The inelastic and elastic failure lines for isothermal creep-fatigue cycles are parallel to the corresponding failure lines for pure fatigue (*pp* cycles).

3. The stress process has a zero mean. The final computed life should be adjusted to account for any mean stress effects.

• The Total-Strain-Range Version of Strain Range Partitioning (TS-SRP) Approach.

1. The four generic Strain Range Partitioning (SRP) life relations are

$$\Delta\epsilon_{in} = C_{ij}(N_{ij})^c \quad (1)$$

$ij = pp, cp, pc, cc$ and c is a constant due to the assumption 2.

2. The interaction damage rule (IDR) is written as follows :

$$\sum \left(\frac{F_{ij}}{N_{ij}} \right) = \frac{1}{N} \quad (2)$$

where F_{ij} is the strain fraction of the type of cycle and is defined as follows [2]:

$$\begin{aligned} F_{pp} &= (\Delta\epsilon_{in})_{pp} / \Delta\epsilon_{in} \\ F_{cp} &= (\Delta\epsilon_{in})_{cp} / \Delta\epsilon_{in} \\ F_{pc} &= (\Delta\epsilon_{in})_{pc} / \Delta\epsilon_{in} \\ F_{cc} &= (\Delta\epsilon_{in})_{cc} / \Delta\epsilon_{in} \end{aligned} \quad (3)$$

with

$$\Delta\epsilon_{in} = (\Delta\epsilon_{in})_{pp} + (\Delta\epsilon_{in})_{cp} + (\Delta\epsilon_{in})_{pc} + (\Delta\epsilon_{in})_{cc}$$

3. Total-strain-range-versus-life relation

$$\Delta\epsilon_t = B(N)^b + C'(N)^c \quad (4)$$

where $\Delta\epsilon_t$ is the total strain-range. Parameters in this equation relate the failure behavior and flow behavior in the following manner:

Failure behavior

$$\Delta\epsilon_{el} = B(N)^b \quad (5)$$

$$\Delta\epsilon_{in} = C'(N)^c \quad (6)$$

where

$$C' = \left[\sum F_{ij}(C_{ij})^{1/c} \right]^c \quad (7)$$

Flow behavior

$$\Delta\epsilon_{el} = K_{ij}(\Delta\epsilon_{in})^n \quad (8)$$

From equations (5), (6), and (8), a desired equation relating flow and failure characteristics is obtained,

$$B = K_{ij}(C')^n \quad (9)$$

• Life Prediction

There are three variants suggested by Saltsman and Halford [1,3] for life prediction by using the total-strain-range version of strain range partitioning (TS-SRP) approach. These variants are summarized as follows.

Variant 1

- (1) Determine the SRP inelastic-strain-range-versus-life relations and the pure fatigue (pp) elastic-strain-range-versus-life relation from failure tests.
- (2) Calculate the cyclic strain-hardening coefficient (K_{ij}) and the strain fractions (F_{ij}) by using an appropriate constitutive flow model, e.g., Walker model [4], for which the material constants are known.
- (3) The elastic line intercept B can now be calculated by using equation (9) and the proceeding information.
- (4) Determine the total-strain-range-versus-life curve for the case in question. Enter the curve at the appropriate total strain range then determine cyclic life for the zero-mean-stress condition. The inversion method of Manson and Muralidharan[5] which is described in section L-2 can also be used to compute the cyclic life N with a given total-strain-range.

Variant 2

- (1) Same as step of variant 1.
- (2) Determine the elastic line intercept B by using the empirical equation of Halford and Saltsman [6]. The constants in this equation are determined from failure data. Failure tests should

be performed at the lower strain ranges to reduce extrapolation errors.

- (3) Measure strain range (elastic and inelastic) and stresses from failure tests and extrapolate to lower strain ranges using empirical equations.
- (4) Determine cyclic life using step 4 of variant 1.

Variant 3

- (1) Same as step 1 of variant 1
- (2) Conduct flow tests for creep-fatigue cycles of interest and obtain from these data necessary empirical correlations describing the flow behavior.
- (3) Calculate the elastic line intercept B by using equation (9). The strain-hardening coefficient K_{ij} and the strain fractions F_{ij} are determined from the correlations obtained from step 2.
- (4) Determine the cyclic life using step 4 of variant 1.

• Mean Stress Consideration [7]:

1. A method for accounting for mean stress effects on life for isothermal conditions has been proposed [8]. The predicted cyclic life accounting for mean stress effects is

$$(N_m)^b = (N)^b - V_{eff} \quad (10)$$

where V_{eff} is the effective mean stress correction term. For isothermal fatigue, V_{eff} is determined by the following equation:

$$V_{eff} = \frac{\sigma_m}{\sigma_a} \exp[-70(\Delta\epsilon_{in}/\Delta\epsilon_{el})^2] \quad (11)$$

where σ_m is the mean stress and σ_a is the stress amplitude. Note that this method was developed for a specific nickel-base alloy and may not apply to other alloys or even to other nickel-base alloys.

2. For thermomechanical fatigue (TMF), an alternate definition of V_{eff} is in order since a mean stress can naturally develop because of the temperature dependency of the yield strength in tension and compression. Hence, V_{eff} in equation (10) should be determined [9] by the following:

$$V_{eff} = \frac{1 + (R_\sigma/R_y)}{1 - (R_\sigma/R_y)} \quad (12)$$

where R_σ is equal to $\sigma_{min}/\sigma_{max}$ and R_y is the absolute value of the ratio of the compressive yield strength to the tensile yield strength at their respective maximum and minimum temperature and strainrates in TMF cycle. However, there is no direct experimental verification of this method for accounting for mean stress effects for nonisothermal fatigue.

- Reliability Considerations:

Because uncertainties exists in the design factor associated with life prediction [10], reliability analysis is a must in life prediction.

The event of failure is

1. $[\Delta\epsilon > \Delta\epsilon_t]$, where $\Delta\epsilon$ is the applied strain range after N cycles.
2. $[N < N_0]$, where N_0 is the design life.

The probability of failure is

1. $P_f = \Pr[\Delta\epsilon > \Delta\epsilon_t]$
2. $P_f = \Pr[N < N_0]$

All the parameters associated with life prediction may be random variables. Life prediction can be made using Wu/FPI or AMVFO.

- References :

- [1] Saltsman, J. F. and Halford, G. R., "Procedures for Characterizing an Alloy and Predicting Cyclic Life With the Total Strain Version of Strainrange Partitioning," NASA Technical Memorandum 4102, June 1989.
- [2] Manson, S. S.; Halford, G. R.; and Hirschberg, M. H., "Creep-Fatigue Analysis by Strain-Range Partitioning," Design for Elevated Temperature Environment ASME, 1971, pp.12-28. (NASA TM X-67838)
- [3] Saltsman, J. F. and Halford, G. R., "An Update of the Total-Strain Version of Strainrange Partitioning," Low Cycle Fatigue, ASTM STP 942, 1988, pp. 329-341.
- [4] Saltsman, J. F. and Halford, G. R., "An Update of the Total-Strain Version of SRP," NASA Technical Paper 2499, Oct., 1985.
- [5] Manson, S. S., and Muralidharan, U., "A Single Expression Formula for Inverting Strain-Life and Stress-Strain Relationship," NASA CR-165347, May, 1981.
- [6] Halford, G. R. and Saltsman, J. F. in *International Conference on Advances in Life Prediction Methods*, D. A. Woodford and J. R. Whitehead, Eds., American Society of Mechanical Engineering, New York, 1983, pp.17-26.

- [7] Saltsman, J. F. and Halford, G. R., "Life Prediction of Thermomechanical Fatigue Using Total Strain Version of Strainrange Partitioning (SRP)," NASA Technical Paper 2779, February, 1988.
- [8] Halford, G. R. and Nachtigall, A. J., "Strainrange Partitioning Behavior of An Advanced Gas Turbine Alloy," AFE-1DA. J. Aircr., Vol. 17, No. 8, Aug. 1980, pp. 598-604.
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- [10] Wirsching, P. H. and Wu, Y. T., "Reliability Considerations for the Total Strain Range Version of Strainrange Partitioning," NASA Contract Report 174757, September, 1984.
- [11] Saltsman, J. F. and Halford, G. R., "Strainrange Partitioning- A Total Strain Range Version," NASA TM 83023, Prepared for the International Conference on Advances in Life Prediction sponsored by the American Society of Mechanical Engineers, Albany, New York, April, 1983.
- [12] Berkovits, A. and Nadiv, S., "Relationship Between Fatigue Life in the Creep-Fatigue Region and Stress-Strain Response," NASA Technical Memorandum 100796, Prepared for the VI International Congress on Experimental Mechanics sponsored by the Society of Experimental Mechanics, Portland, Oregon, June, 1988.

Appendix B.2
**Linear Elastic Fracture Mechanics (LEFM) Resistance Models and
Fatigue and Creep Resistance Models**

Rocketdyne Division, Rockwell International

**Linear Elastic Fracture Mechanics
(LEFM)
Resistance Models**

Kaung Jain Chang
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July 5, 1990

FRACTURE MECHANICS (Preliminary)

At Rocketdyne most fracture mechanics analyses employ Linear Elastic Fracture Mechanics (LEFM) principles; and most crack problems are of mode I. The stress intensity factor (K_I) solutions published in the literature and the handbooks (Ref. 1-4) are used. A plastic zone correction is typically included in the stress intensity factor calculations. The most common crack configuration is the semi-elliptical surface crack; in such a case, Newman and Raju's (Ref. 5) solution is used to calculate the K_I values. Furthermore, the crack propagation is two dimensional, it grows both in the depth (a) and in the length (c) directions with its shape varying but remaining elliptical.

Calculation of the crack growth is performed by integrating the fatigue crack growth rate (FCGR) equation, which take the following form

$$\frac{da}{dN} = f(a, \Delta K_I) \quad (1)$$

and the fatigue crack growth (FCG) life of the member (N_f) is

$$N_f = \int_{a_i}^{a_f} \frac{da}{f(a, \Delta K_I)} \quad (2)$$

where a_i and a_f are the initial and final crack depths, respectively.

Two FCGR equations are being used at Rocketdyne to represent the fatigue crack growth rate. These are the Walker equation (Ref. 6) and the modified Forman equation (Refs. 7,8). With further simplification the Walker equation can be integrated to obtain an explicit algebraic expression for the fatigue crack growth life N_f . The modified Forman equation is employed in the NASA-FLAGRO (Ref. 8) computer code, which is an incremental flaw growth code commonly used at Rocketdyne.

In the integration of the FCGR equations, the initial crack depth a_i and the final crack depth a_f of the integration are dependent upon the requirements of the analysis. The initial crack depth a_i can be the NDI capability, an observed crack size, or the crack size defined by proof test screening. The final crack depth a_f can be obtained from the conditions of toughness failure when $K_I = K_C$, where K_C is the material toughness; leakage when $a = t$, where t is the thickness of the member; or the threshold condition when $\Delta K_I = \Delta K_{th}$ or $K_I = K_{th}$, where ΔK_{th} is the FCGR threshold and K_{th} (K_{ISCC}) is the sustained load crack growth threshold of the material.

A simple approximation to the short crack effect using El Haddad's (Ref. 9) intrinsic flaw size parameter can be simply incorporated and is often employed as needed.

NASA-FLAGRO is used regularly in the FCG calculations at Rocketdyne. However, this program is big and contains multiple calculations in addition to the FCG calculation. It is not feasible to adopt such a program in the current PSAM system. A new subprogram should be written for the purpose of simulating incremental flaw growth that is compatible to the PSAM system, for the FCG calculations.

The Walker Equation

The Walker equation (Ref. 6) assumes that the cyclic FCGR (da/dN) equation can be expressed in a power law form

$$\frac{da}{dN} = C \left[\frac{\Delta K_I}{(1-R)^{1-m}} \right]^n \quad (3)$$

where ΔK_I is the stress intensity factor range; R is the R-ratio defined as the ratio of the maximum and minimum stress intensity factors, K_{min}/K_{max} ; and C , m and n are material constants.

Note that when the R-ratio has the value zero, Eq.(3) reduces to the common Paris Equation as follows

$$\frac{da}{dN} = C [\Delta K_I]^n$$

The stress intensity factor K_I has a general form as given below

$$K_I = \lambda \sigma \sqrt{a} \quad (4)$$

Let us assume that the flaw shape does not change as the flaw advances and that the shape to front and back face correction (λ) remains the same as the flaw grows. Based on the above equations, the FCGR integration can be expressed in an explicit algebraic expression.

First, based on the definition of the R-ratio (R), the expression of ΔK_I becomes

$$\Delta K_I = K_{max} (1 - R), \quad (5)$$

and Eq.(3) can be written as

$$\frac{da}{dN} = C \left[\frac{K_{max}(1-R)}{(1-R)^{1-m}} \right]^n = C K_{max}^n (1-R)^{mn} \quad (6)$$

Second, let $C^* = C (1-R)^{mn}$, and utilize Eq.(4), the FCGR equation becomes

$$\frac{da}{dN} = C^* (\lambda \sigma_{max} \sqrt{a})^n \quad (7)$$

Rearranging this equation, one obtains

$$\frac{da}{a^{n/2}} = C^* (\lambda \sigma_{max})^n dN \quad (8)$$

Third, integrating the above equation from an initial flaw size a_i to a final flaw size a_f :

$$\int_{a_i}^{a_f} \frac{da}{a^{n/2}} = \int_0^{N_f} C^* (\lambda \sigma_{max})^n dN \quad (9)$$

The above integration can be obtained explicitly for two cases, when $n \neq 2$ and $n = 2$:

(1) When $n \neq 2$, the number of cycles N_f in terms of the crack sizes is

$$N_f = \left(a_i^{-(n-2)/2} - a_f^{-(n-2)/2} \right) \frac{2}{(n-2)(\lambda \sigma_{max})^n C^*} \quad (10)$$

or in terms of the stress intensity factors,

$$N_f = \left(K_{max,i}^{(2-n)} - K_{max,f}^{(2-n)} \right) \frac{2}{(n-2)(\lambda \sigma_{max})^2 C^*} \quad (11)$$

(2) When $n = 2$, Eq.(9) becomes

$$\int_{a_i}^{a_f} \frac{da}{a} = \int_0^{N_f} C^* (\lambda \sigma_{max})^2 dN \quad (12)$$

Then the solution N_f can be obtained as

$$N_f = \frac{\ln \left(\frac{a_f}{a_i} \right)}{C^* (\lambda \sigma_{max})^2} \quad (13)$$

The Modified Forman Equation

Forman's modification of the Paris equation accounts for instability when the maximum stress intensity approaches the toughness value K_C and crack arrest when the range of stress intensity is below ΔK_{th} . It is written in the form

$$\frac{da}{dN} = \frac{C(1-R)^m \Delta K_I^n [\Delta K_I - \Delta K_{th}]^p}{[(1-R)K_C - \Delta K_I]^q} \quad (14)$$

where C , m , n , p , and q are material constants, and ΔK_{th} is the FCGR threshold.

Usage of this relationship will require the implementation of a numerical integration scheme for life prediction. Additional complexity could be included by anticipating the flaw shape changes pattern as the crack advances under certain crack geometries and applied loadings.

Spectrum Loading Considerations

For the case of a spectrum loading, a Miner's type linear damage rule is assumed during the FCG calculations. Two ways of handling the spectrum loading are used at Rocketdyne. First, the stress ranges are collectively rearranged into a number of load steps using a histogram; and then the different steps of loads are specified as input loads to a code such as NASA-FLAGRO to calculate the FCG.

Second, an equivalent stress range technique is used in conjunction with the Walker equation to convert the spectrum into an equivalent single stress range using the material $da/dN - \Delta K_I$ data. In this approach, a nominal stress range is assumed as the reference load, and each of the other stress ranges of the spectrum is converted to an equivalent number of cycles of the reference stress range that would have the same FCGR as if not being converted. The sum of all the equivalent reference stress range cycles provides the number of cycles of the reference stress range that is equivalent to the loading spectrum.

In applying the equivalent stress range technique, the load spectrum need to be screened constantly to identify the stress ranges that the ΔK_I values fall below the threshold range. The number of such load steps are changing as the crack grows, because the K_I amplitudes are changing as the crack size changes. Cycles that the ΔK_I values fall below the threshold range do not contribute to crack growth. However, assuming the flaw shape (a/c) and geometry factor (λ) are invariant with the crack size, the threshold flaw size a_{th} can be predetermined for each step in the spectrum. This simply and conveniently identifies the crack growth intervals where distinct equivalent numbers of cycles of the reference stress range must be determined.

The Random Variable Candidates

In the FCG calculations, the initial crack size a_i , the Toughness K_C , the threshold ΔK_{th} or K_{th} are random variables. The initial crack size distribution can present a pre-existing flaw population suitably modified by NDE or proof test screening. The crack shape or aspect ratio a/c where c is the half crack length is also a random variable to define the randomness of the two dimensional surface crack shape. Furthermore, the material constants introduced in the FCGR equations, such as C , n , and m in the Walker equation, and C , n , m , p , and q in the modified Forman's equation, are potential random variables. It is expected that n in the Walker equation and n , p , and q in the modified Forman equation would be deterministic variables.

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Fatigue and Creep Resistance Models

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1 Introduction

This write-up gives a short summary of the fatigue and creep resistance models used at Rocketdyne. Several approaches with varying degree of sophistication are in use. All the presented approaches have their drawbacks. It is not the purpose of this write-up to present the most 'accurate' model but to present those commonly used at Rocketdyne.

A fatigue analysis normally proceeds along the following points:

- (A) Define component geometry and material properties
- (B) Perform global structural analysis
- (C) Define stress history
- (D) Count cycles
- (E) Perform notch analysis
- (F) Calculate damage for entire load history
- (G) Determine life

The first two items listed above are not directly addressed here.

2 Stress History

2.1 General

Analyses of rocket engine systems and components can be performed for transient and steady state conditions. The conditions during transients typically produce only a few cycles of higher load level responses and, therefore, affect only the low cycle fatigue damage. The following discussion will be limited to steady state since most analysis are done for that condition. Furthermore, the steady state condition at maximum power level is normally considered although SSME, for example, can operate at several power levels.

The load history at steady state is usually composed of several sinusoidal loads vibrating each at their own frequency and random loads as defined by a power spectral density function (PSD) superimposed onto the steady state load level. The sinusoidal loads are often due to rotating machinery while the random loads are normally due to combustion processes. In a typical deterministic analysis each of three orthogonal axes are analyzed separately.

The dynamic stresses at a point can be defined by the root-sum-square of the random stress component j , σ_j^r , the sinusoidal *amplitude* of each stress component j for each exciting frequency n , $\sigma_{j,n}^{ss}$, and the expected frequency, f_{exp}^c of the combined dynamic load history (normally based on effective stress and effective stress rate). There are n_{ss} exciting frequencies and only one source for the random loads. The details of how these values are commonly calculated at Rocketdyne are given in reference [1].

An alternate and more precise way to define the dynamic stresses is to specify a PSD of the random stress components at the location of interest together with the amplitudes of the sinusoidal stress components, $\sigma_{j,n}^{ss}$ and its frequency, f_n^{ss} .

The following sections show how the dynamic load history is simplified for a fatigue damage calculation at Rocketdyne.

2.2 Sinusoidal Plus 3σ Value Random Loads

The complex dynamic stress history is simply replaced by a sinusoidal stress history with the previously mentioned expected frequency f_{exp}^c and a constant combined amplitude of

$$\sigma_j^c = 3\sigma_j^r + \sum_{n=1}^{n_{ss}} \sigma_{j,n}^{ss} . \quad (1)$$

The summation should be made at the stress *component* level so that the proper effective stress of the stress range can be calculated.

For details on combining the dynamic loads refer to [1].

The combined expected frequency, f_{exp}^c is based on the number of zero crossings and, therefore, can ignore some cycles which do not cross zero but nevertheless do damage to the material. Furthermore, the 'local' mean stress and strain of a dynamic cycle is ignored in the calculation of the fatigue damage only the 'global' mean stress and strain associated with the steady state mean load level can be considered. Figure 1 shows a stress vs. time history and indicates the 'global' and 'local' mean stress. A similar definition applies to the 'global' and 'local' mean strain. The above mentioned zero crossing occurs when the stress vs. time trace crosses the 'global' mean stress in figure 1.

2.3 Sinusoidal Plus Rayleigh Distributed Random Loads

The peaks of the random load is assumed to follow the Rayleigh distribution with the probability density function

$$f(x) = \begin{cases} \frac{x}{\sigma^2} e^{-\frac{1}{2}(\frac{x}{\sigma})^2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases} \quad (2)$$

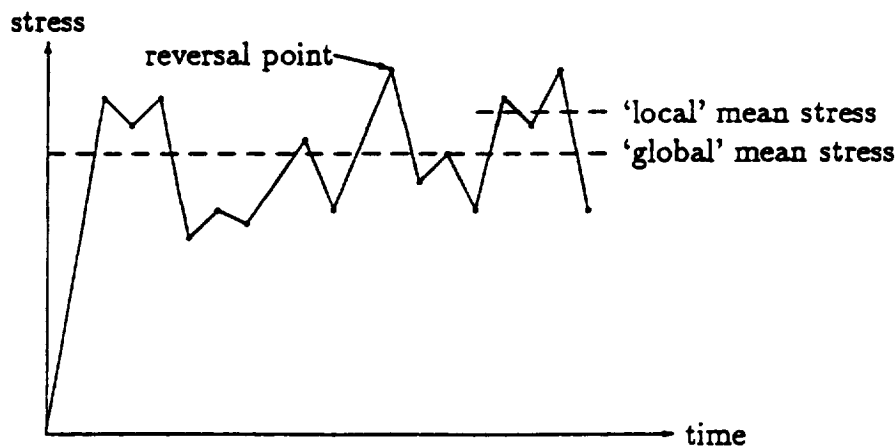


Figure 1: Typical stress vs. time history showing the 'global' mean stress of the whole stress history and the 'local' mean stress of one cycle.

The σ in the above equation is the parameter of the Rayleigh distribution and equals the previously mentioned root-sum-square of the random stress component j , σ_j^r . The sinusoidal load is superimposed assuming the frequency and the phase angles are all identical. Note that the sinusoidal part of the combined stress amplitude is regarded to be a fixed deterministic value while the random part follows a probabilistic distribution. The combined amplitude for the stress component j is written as

$$\sigma_j^c = \lambda \sigma_j^r + \sum_{n=1}^{n_{ss}} \sigma_{j,n}^{ss} \quad \lambda \geq 0 \quad (3)$$

with λ being a parameter specifying the magnitude of the combined stress amplitude. Let σ_j^a be defined as

$$\sigma_j^a = \sigma_j^c - \sum_{n=1}^{n_{ss}} \sigma_{j,n}^{ss}$$

then the resulting probability density function of the combined amplitude becomes

$$f(\sigma_j^c) = \begin{cases} \frac{\sigma_j^a}{(\sigma_j^r)^2} e^{-\frac{1}{2}(\frac{\sigma_j^a}{\sigma_j^r})^2} & \text{if } \sigma_j^c > \sum_{n=1}^{n_{ss}} \sigma_{j,n}^{ss} \\ 0 & \text{if } \sigma_j^c \leq \sum_{n=1}^{n_{ss}} \sigma_{j,n}^{ss} \end{cases} \quad (4)$$

Figure 2 shows the resulting probability density function for the combined alternating amplitude. From that figure it can be seen that the sinusoidal part of the alternating

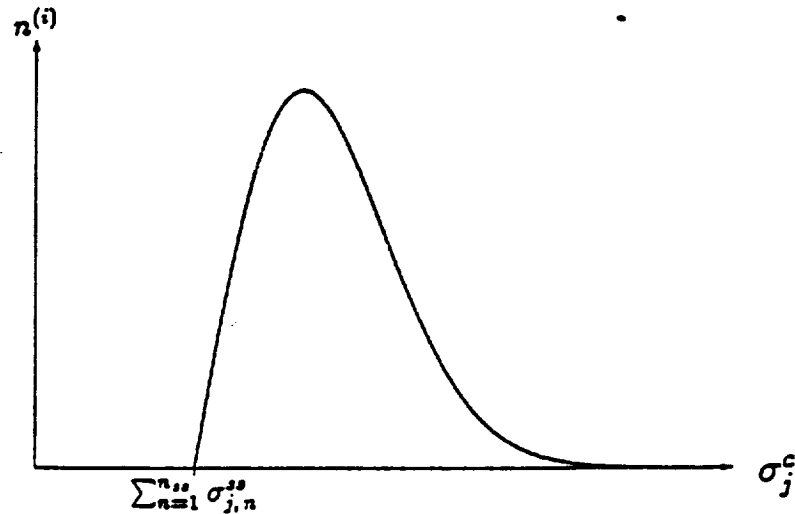


Figure 2: Simplified probability density function for the sinusoidal plus Rayleigh distributed random loads.

stress simply causes a shift to the right of the Rayleigh distribution¹.

These combined amplitudes are again assumed to oscillate at the previously mentioned expected frequency, f_{exp}^c .

The same remarks about the expected frequency and the 'local' mean stress and strain as mentioned in the previous section apply here.

2.4 Stress vs. Time History

This approach uses the PSD of the random stress components and the set of sinusoidal stress components, $\sigma_{j,n}^{ss}$ with its frequencies, f_n^{ss} to create an actual stress vs. time history at the location of interest. The PSD of the random stress component is discretized so that the power spectrum in a small frequency window is replaced by an equivalent sine function with a constant amplitude, $\sigma_{j,m}^{random}$ and frequency, f_m^{random} . Thus, the whole PSD of the random load is replaced by n_{random} sine functions with amplitude and frequency as derived from the PSD. A sample of the actual time history is now generated by combining the various sine functions given by the PSD with all the initially given sinusoidal amplitudes

$$\begin{aligned} \sigma_j^c(t) = & \sum_{n=1}^{n_{ss}} \left[\sigma_{j,n}^{ss} \sin \left(\frac{2\pi t}{f_n^{ss}} + \phi_n^{ss} \right) \right] + \dots \\ & \dots + \sum_{m=1}^{n_{random}} \left[\sigma_{j,m}^{random} \sin \left(\frac{2\pi t}{f_m^{random}} + \phi_m^{random} \right) \right]. \end{aligned} \quad (5)$$

The phase angles ϕ_n^{ss} and ϕ_m^{random} have to be randomized² to arrive at one sample history. Several time history realizations exist for one given PSD and one set of sinusoidal amplitudes. Therefore, a Monte Carlo simulation which includes the fatigue and creep damage calculation has to be performed to account for the variations in those sample load histories.

This method certainly requires the highest computational effort of all the three presented approaches but removes the uncertainty associated with the use of the expected frequency, f_{exp}^c . Proper cycle counting routines can also determine the 'local' mean stress and strain for subsequent consideration during the fatigue damage calculations.

A typical Space Shuttle Main Engine flight duration is approximately 500 seconds. It is often not necessary to create a stress vs. time history for the whole 500 seconds to

¹This shift results from the assumption that the sinusoidal load is in-phase with the random loads and oscillates at the same frequency. A correct superposition of the random and sinusoidal loads would result in a distortion of the original Rayleigh distribution.

²A uniform distribution between the limits 0 and 2π is used for randomizing each phase angle independently.

calculate the high cycle fatigue damage. A stress vs. time history of just a couple seconds might be regarded as long enough to represent the appropriate fraction of the whole 500 seconds history. Significant savings in computations can be achieved if this shorter stress history is used.

3 Cycle Counting

If the stress history is defined as described in section 2.2 or 2.3 then there is no need for counting cycles. The number of dynamic cycles is defined by the combined expected frequency f_{exp}^c , the duration of the stress history and the steady state stress cycle (start/stop cycle).

Cycle counting is necessary if the stress history is defined as stress vs. time. The preferred method at Rocketdyne is the 'Loop Closure Method' as described to some detail in reference [2] and in the example given in section 6.2 (page 21).

The tracking of each closed stress-strain loop requires a large computer memory and considerable computing time. Simplifications to the cycle counting routine can be made if the effect of the 'local' mean strain (see figure 1) to the fatigue damage is ignored for the high cycle fatigue³ cycles. Ignoring the 'local' mean strain can provide a reasonable approximation to the high cycle fatigue damage if the 'global' mean strain is relatively large compared to the 'local' mean strain. In this case a 'Half Cycle Counting Method' can be implemented. This method identifies each half cycle between two reversal points and calculates the fatigue damage due to that one reversal. The method is easy to implement since the cycle counting simply proceeds from one reversal point to the next. If the fatigue damage from one half cycle is calculated that half cycle will not be required anymore and, therefore, it does not have to be stored.

³The dynamic loads during steady state are commonly referred to as high cycle fatigue loads and the loads during engine start or stop transients are usually called low cycle fatigue loads at Rocketdyne.

4 Notch Analysis

4.1 Neuber's Rule

Cracks preferably initiate at stress concentrations like notches. A coarse grid finite element model does not pick up the full effect of the stress concentration and it is in practice difficult to know how much the finite element model results are affected by the stress concentration. It is a common practice at Rocketdyne to conservatively assume that a coarse grid finite element model does not include the effect of any stress concentration.

Several approximate analysis methods exist to estimate the elastic-plastic stress and strain at the notch root (surface) if the notch plasticity can be categorized as contained plasticity. The most often used method is Neuber's rule.

The elastic nominal stress component j at some time t_1 when a stress reversal occurs in the stress history is designated $\sigma_j^{el,nom}(t_1)$ (it takes into account the steady state mean stress). At time $t_2 > t_1$ the next stress reversal occurs and the elastic nominal stress component j has the value $\sigma_j^{el,nom}(t_2)$. Therefore, the elastic nominal stress excursion is

$$\Delta\sigma_j^{el,nom} = \sigma_j^{el,nom}(t_2) - \sigma_j^{el,nom}(t_1). \quad (6)$$

The elastic stress concentration factor $K_{t,j}$ acts on the stress component j . Therefore, the elastic stress range of component j at the notch root is

$$\Delta\sigma_j^{el} = K_{t,j}\Delta\sigma_j^{el,nom}. \quad (7)$$

Now, the elastic effective stress range $\Delta\sigma_{eff}^{el}$ can be calculated based on $\Delta\sigma_j^{el}$ using von Mises' yield criteria

$$\begin{aligned} \Delta\sigma_{eff}^{el} = & \left[\frac{1}{2} \left\{ (\Delta\sigma_{xx}^{el} - \Delta\sigma_{yy}^{el})^2 + (\Delta\sigma_{yy}^{el} - \Delta\sigma_{zz}^{el})^2 + (\Delta\sigma_{zz}^{el} - \Delta\sigma_{xx}^{el})^2 \right. \right. \\ & \left. \left. + 6 \left[(\Delta\sigma_{xy}^{el})^2 + (\Delta\sigma_{yz}^{el})^2 + (\Delta\sigma_{zx}^{el})^2 \right] \right\} \right]^{1/2}. \end{aligned} \quad (8)$$

Neuber's rule states that

$$\frac{(\Delta\sigma_{eff}^{el})^2}{E} = \Delta\sigma_{eff}^{el-pl} \Delta\epsilon_{eff}^{el-pl} = \Delta\sigma_{eff}^{el-pl} 2 g \left(\frac{\Delta\sigma_{eff}^{el-pl}}{2} \right) \quad (9)$$

where

$$\epsilon_{eff}^{el-pl} = g(\sigma_{eff}^{el-pl}) \quad (10)$$

describes the cyclic stabilized stress-strain behavior of the material and the factors of 2 result from the application of Masing's hypothesis for a stabilized hysteresis loop.⁴

The elastic-plastic effective stress $\Delta\sigma_{eff}^{el-pl}$ can be calculated from equation 9 and the resulting elastic-plastic strain $\Delta\epsilon_{eff}^{el-pl}$ from equation 10. The stress-strain behavior is often modeled by a bilinear curve or a Ramberg-Osgood equation.

4.2 Fatigue Notch Factor

The total fatigue life is commonly defined to be the sum of the crack initiation life and the crack propagation life. While the crack initiation life⁵ is quite well characterized by the local stress and strain, the crack propagation life is significantly dependent on stress gradients along the crack path. Since most fatigue tests are performed on smooth specimens and provide the total fatigue life the difference in crack propagation behavior of a smooth specimen and a notched specimen has to be taken into account by a factor. This factor is called the fatigue notch factor, K_f .

It is important to realize that if only crack initiation life is of interest and the test data is collected for crack initiation then the fatigue notch factor, K_f is practically equal to the stress concentration factor, K_t for linear elastic stresses (elastic-plastic stresses can be handled with Neuber's rule). However, if the total fatigue life is of interest then the fatigue notch factor, K_f is always equal or less than the stress concentration factor, K_t (again for linear elastic stresses).

The fatigue notch factor, K_f can be estimated based on fracture mechanics considerations since it mainly depends on the difference in crack propagation. This is described in reference [3] for the linear elastic case.

The fatigue notch factor, K_f is a function of notch geometry, material properties, applied stress ratio R , and number of cycles to failure N_f . However, if the fatigue notch

⁴Equation 9 is sometimes written as

$$\frac{2(1+\nu)}{3} \frac{(\Delta\sigma_{eff}^{el})^2}{E} = \Delta\sigma_{eff}^{el-pl} \Delta\epsilon_{eff}^{el-pl}$$

which reflects the relation between effective stress and effective strain. This form is not used in this write-up.

⁵The stages of crack initiation and crack propagation are difficult to distinguish on theoretical grounds. In practice, however, the crack initiation phase can be defined as the forming of a crack with a fixed size which can be tied, for example, to microstructural features of the material. Because of the controversial nature of this subject several definitions besides the one mentioned can be found in the literature.

factor is desired at the endurance limit⁶, K_f^{end} then simple expressions are available. These expressions can be based either on a fatigue crack propagation threshold (see for example [3]) or on an empirical equation developed by Peterson [4, pages 9 to 11]. Here, only the latter is going to be shown:

$$K_f^{end} = 1 + \frac{K_t - 1}{1 + \frac{\alpha}{r}} \quad (11)$$

where K_f^{end} = fatigue notch factor at the endurance limit,
 K_t = theoretical, linear elastic stress concentration factor,
 α = material constant depending on strength and ductility,
 r = notch root radius.

Some average values of α for aluminum and steel are given in reference [5].

If fatigue test data are available for the notch size and operating condition of interest it is strongly recommended to use those data directly.

If a fatigue notch factor is to be taken into account in the fatigue analysis then the $K_{t,j}$ in equation 7 has to be replaced by a $K_{f,j}^{end}$ (the fatigue notch factor acting on stress component j)

$$\Delta\sigma_{j,K_f}^{el} = K_{f,j}^{end} \Delta\sigma_j^{el,nom} \quad (12)$$

and the effective stress range is calculated as in equation 8 but with the above stress ranges.

$$\begin{aligned} \Delta\sigma_{eff,K_f}^{el} = & \left[\frac{1}{2} \left\{ \left(\Delta\sigma_{xx,K_f}^{el} - \Delta\sigma_{yy,K_f}^{el} \right)^2 + \right. \right. \\ & \left. \left(\Delta\sigma_{yy,K_f}^{el} - \Delta\sigma_{zz,K_f}^{el} \right)^2 + \left(\Delta\sigma_{zz,K_f}^{el} - \Delta\sigma_{xx,K_f}^{el} \right)^2 + \right. \\ & \left. \left. 6 \left[\left(\Delta\sigma_{xy,K_f}^{el} \right)^2 + \left(\Delta\sigma_{yz,K_f}^{el} \right)^2 + \left(\Delta\sigma_{zx,K_f}^{el} \right)^2 \right] \right\} \right]^{1/2} . \quad (13) \end{aligned}$$

Note that the elastic effective stress range should be below the yield strength of the material since the fatigue notch factor at the endurance limit is used!

As will be discussed later, the fatigue damage is not only dependent on the applied stress range but also on the mean stress and mean strain. In practice, it is easier to define the fatigue notch factor in such a way that the *actual* local mean stress (and/or strain) at the notch root is used. In other words, the fatigue notch factor is applied to the alternating stress (or strain range) while the appropriate stress (or strain) concentration

⁶It is Rocketdyne's practice to define the endurance limit as the stress amplitude at 10^7 cycles in a S-N curve even if the curve does not exhibit a plateau there.

factor is applied to the mean stress (or strain). This definition is used in the remaining discussion.

It is again emphasized that $K_{f,j}$ depends on the number of constant amplitude cycles to failure N_f among other things. This cycle dependency is in general not known. Therefore, if a variable amplitude load history — for which each amplitude level occurring in the history results in a different number of cycles to failure — is considered the application of $K_{f,j}$ becomes difficult.

5 Damage Accumulation

5.1 'Conventional' Method

For those components experiencing high cycle and low cycle fatigue damage as well as creep damage, the following generalized life equation is commonly used at Rocketdyne:

$$\Phi_{tot} = \Phi_{fH} + 4\Phi_{fL} + 4\Phi_c \quad (14)$$

where Φ_{tot} = total damage,
 Φ_{fH} = high cycle fatigue damage,
 Φ_{fL} = low cycle fatigue damage,
 Φ_c = creep damage.

The total damage Φ_{tot} is assumed to equal 1 at failure although experimentally obtained values can vary from .25 to 3. The factors of 4 in the above equation are safety factors; a safety factor on the equivalent alternating stress, $\sigma_{eq alt}$ is applied in the case of high cycle fatigue (see equation 16).

In general, the effective stresses (mean and alternating) are used in the damage calculation for high cycle fatigue. However, when the alternating stresses in two perpendicular directions have the same sign and are in phase, the maximum principal stresses (mean and alternating) may be used. The effective stresses are used in the following discussion.

5.1.1 High Cycle Fatigue

The high cycle fatigue damage is evaluated by a linear damage rule (Miner's rule)

$$\Phi_{fH} = \sum_{i=1}^{n_{HCF}} \frac{n^{(i)}}{N_f^{(i)}} \quad (15)$$

where $n^{(i)}$ = the actual number of cycles at a particular equivalent alternating stress amplitude, $\sigma_{eq alt}^{(i)}$,

$N_f^{(i)}$ = the cycles to failure at that stress amplitude $\sigma_{eq alt}^{(i)}$,

n_{HCF} = the number of different stress amplitudes $\sigma_{eq alt}^{(i)}$ in the stress history.

The cycles to failure at the equivalent alternating stress amplitude can be read from stress vs. life (S-N) material curves.

The equivalent alternating stress takes into account the effect of the mean stress. If test data is available then it should be used otherwise a modified Goodman rule is used. The rule is represented by the formula

$$\sigma_{eq\ alt}^{(i)} = (FS)_{eq\ alt} \frac{\sigma_{alt}^{(i)}}{1 - \frac{(\sigma'_{mean})^{(i)}}{F_{tu}}} \quad (16)$$

where $\sigma_{eq\ alt}^{(i)}$ = equivalent alternating stress amplitude at a stress ratio $R = -1$,
 $(FS)_{eq\ alt}$ = factor of safety on the equivalent alternating stress amplitude:

$$(FS)_{eq\ alt} = \begin{cases} 1.25 & \text{for stationary components,} \\ 1.40 & \text{for rotational components,} \end{cases}$$

$\sigma_{alt}^{(i)}$ = alternating stress amplitude (includes the effect of $K_{f,j}$),
 $\sigma_{mean}^{(i)}$ = actual mean stress (includes the effect of $K_{t,j}$),
 F_{tu} = ultimate strength,
 F_{ty} = yield strength,
 $(\sigma'_{mean})^{(i)}$ = adjusted mean stress:

$$(\sigma'_{mean})^{(i)} = \begin{cases} \sigma_{mean}^{(i)} & \text{if } \sigma_{alt}^{(i)} + \sigma_{mean}^{(i)} \leq F_{ty}, \\ F_{ty} - \sigma_{alt}^{(i)} & \text{if } \sigma_{alt}^{(i)} + \sigma_{mean}^{(i)} > F_{ty}, \\ 0 & \text{if } \sigma_{alt}^{(i)} > F_{ty}. \end{cases} \quad (17)$$

The above equation is based on the assumption of an elastic-perfectly plastic material behavior.

5.1.2 Low Cycle Fatigue

The low cycle fatigue damage is also evaluated by a linear damage rule

$$\Phi_{fL} = \sum_{i=1}^{n_{LCF}} \frac{n^{(i)}}{N_f^{(i)}} \quad (18)$$

where $n^{(i)}$ = the actual number of cycles at a particular equivalent strain range, $\Delta\epsilon_{eq}^{(i)}$,
 $N_f^{(i)}$ = the cycles to failure at that equivalent strain range, $\Delta\epsilon_{eq}^{(i)}$,
 n_{LCF} = the number of different strain ranges $\Delta\epsilon_{eq}^{(i)}$ in the stress history.

The cycles to failure at the equivalent strain range can be read from strain vs. life material curves.

The equivalent strain range takes into account the effect of the mean stress and strain. In practice this effect is often ignored but approximations can be made as follows (if test data is available on this effect then it should be used). Manson's Method of Universal Slopes equation [6] modified for mean stress and mean strain by R. Cooper and discussed with S. S. Manson is

$$\begin{aligned}\Delta\epsilon_{tot} &= \Delta\epsilon_{el} + \Delta\epsilon_{pl} \\ &= \frac{3.5(F_{tu} - \sigma_{mean})}{E} N_f^{-.12} + (D - |\epsilon_{mean}|)^{-.6} N_f^{-.6}\end{aligned}\quad (19)$$

where $\Delta\epsilon_{tot}$ = total strain range,

$\Delta\epsilon_{el}$ = elastic part of the total strain range,

$\Delta\epsilon_{pl}$ = plastic part of the total strain range,

D = material ductility ($D = \ln\left(\frac{100}{100-R.A.}\right)$ and $R.A.$ is the reduction of area in percent),

ϵ_{mean} = mean strain.

If the effect of the mean stress is ignored on the low cycle fatigue life and only the mean strain is considered then

$$\Delta\epsilon_{pl} = \Delta\epsilon_{tot} - \frac{\Delta\sigma}{E} = (D - |\epsilon_{mean}|)^{-.6} N_f^{-.6}\quad (20)$$

and a simple expression for the equivalent strain range can be obtained

$$\Delta\epsilon_{eq}^{(i)} = \Delta\epsilon_{el}^{(i)} + \frac{\Delta\epsilon_{pl}^{(i)}}{\left(1 - \frac{|\epsilon_{mean}^{(i)}|}{D}\right)^{-.6}}\quad (21)$$

Note that the form of the above equation is very similar to the Goodman rule used in high cycle fatigue.

5.1.3 Creep Rupture

The creep rupture damage is again evaluated by a linear damage rule

$$\Phi_c = \sum_{i=1}^{n_{creep}} \frac{t^{(i)}}{t_r^{(i)}}\quad (22)$$

where $t^{(i)}$ = the actual time at a particular hold-time stress level, $\sigma_{hold}^{(i)}$
 $t_r^{(i)}$ = the time to rupture at that hold-time stress, $\sigma_{hold}^{(i)}$

n_{creep} = the number of different hold-time stresses in the stress history.

The time to rupture at the hold-time stress, $t_r^{(i)}$, are obtained from stress-rupture life curves, which are experimentally determined for the materials and temperatures of interest. Empirical equations [8] for the stress-rupture life curves are usually written in the following form

$$\log t_r = c + P(\sigma, T) \quad (23)$$

where $P(\sigma, T)$ has been proposed in a number of different forms:

$$(1) P(\sigma, T) = b_1/T + b_2 X/T + b_3 X^2/T + b_4 X^3/T$$

$$(2) P(\sigma, T) = b_1/T + b_2 X + b_3 X^2 + b_4 X^3$$

$$(3) P(\sigma, T) = (T - T_a)(b_1 + b_2 X + b_3 X^2 + b_4 X^3)$$

These are the Larson-Miller [9], Sherby-Dorn [10], and Manson-Haferd [11] relationships, respectively. In the above expressions,

t_r = the time in hours to rupture at a stress level σ

c = the regression constant

b_i = the coefficients (b_1 through b_4)

T = the absolute temperature (T_a is the temperature of convergence of the iso-stress lines)

X = the log stress, $\log(\sigma)$.

While all the forms may be used to model a data set with varying degrees of goodness of fit, experience and practice indicate the Larson-Miller relationship adequately models most materials and is usually the preferred equation form. If none of these standard forms satisfactorily follow the data trends, various other combinations of stress and temperature may be tried.

5.2 Total Fatigue Curve

The interaction of fatigue and creep damage is accounted for in a similar way as before

$$\Phi_{tot} = \Phi_{fatigue} + \Phi_c \quad (24)$$

where Φ_{tot} = total damage,

$\Phi_{fatigue}$ = fatigue damage,

Φ_c = creep rupture damage.

The creep damage is calculated as described in section 5.1.3 and the total damage Φ_{tot} is assumed to equal 1 at failure as mentioned in section 5.1.

The fatigue damage is determined from a total fatigue curve as proposed by Morrow

$$\begin{aligned}\frac{\Delta\epsilon_{tot}}{2} &= \frac{\Delta\epsilon_{el}}{2} + \frac{\Delta\epsilon_{pl}}{2} \\ &= \frac{\sigma_f}{E} (2N_f)^b \left(1 - \frac{\sigma_{mean}}{F_{tu}}\right) + \epsilon_f (2N_f)^c \left(1 - \frac{|\epsilon_{mean}|}{D}\right)\end{aligned}\quad (25)$$

and the consistent Ramberg-Osgood type of stress-strain curve

$$\frac{\Delta\epsilon_{tot}}{2} = \frac{\Delta\epsilon_{el}}{2} + \frac{\Delta\epsilon_{pl}}{2} = \frac{\Delta\sigma}{2E} + \epsilon_f \left(\frac{\Delta\sigma}{2\sigma_f}\right)^{c/b}\quad (26)$$

where $\Delta\sigma$ = stress range,

$\Delta\epsilon_{tot}$ = total strain range,

$\Delta\epsilon_{el}$ = elastic part of the total strain range,

$\Delta\epsilon_{pl}$ = plastic part of the total strain range,

N_f = number of cycles to failure ($2N_f$ is the number of reversals to failure),

E = Young's modulus,

$\sigma_f, \epsilon_f, b, c$ = curve fitting parameters.

Knowing the effective stress range, $\Delta\sigma^{(i)}$, the elastic and plastic strain ranges, $\Delta\epsilon_{el}^{(i)}$ and $\Delta\epsilon_{pl}^{(i)}$, can be calculated from equation 26. Then, the fatigue damage for all fatigue cycles, n_{fat} , can be derived from equation 25

$$\Phi_{fatigue} = \sum_{i=1}^{n_{fat}} \frac{1}{N_f^{(i)}} = 2 \sum_{i=1}^{n_{fat}} \left[\frac{\sigma_f}{E\epsilon_f} \left(\frac{1 - \sigma_{mean}^{(i)}/F_{tu}}{1 - |\epsilon_{mean}^{(i)}|/D} \right) \frac{\Delta\epsilon_{pl}^{(i)}}{\Delta\epsilon_{el}^{(i)}} \right]^{\frac{1}{1-c}} \quad (27)$$

6 Examples Using Deterministic Analysis Procedures

6.1 Example I: 'Conventional' Method

The following example details the analysis procedure for a case when the stress history is given as a superposition of sinusoidal loads plus a Rayleigh distributed random load and the damage is calculated based on the 'conventional' method. The sinusoidal plus random loads are associated with high cycle fatigue damage and are assumed to result in elastic strains only while the start/stop loads are associated with low cycle fatigue and creep damage and can cause a locally elastic-plastic response.

For the remainder of the text the following abbreviation is used to specify the calculation of the effective stress from the component stresses:

$$\begin{aligned}\sigma_{eff} &= \Psi(\sigma_j) \\ &= \sqrt{\frac{1}{2} \left\{ (\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6 [\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2] \right\}}.\end{aligned}$$

(A) Component Geometry and Material Properties:

The following parameters are given:

- Geometry: $K_{t,j}$ = elastic stress concentration factor acting on stress component j ,
 $K_{f,j}$ = fatigue notch factor (assumed to be constant here) for stress component j ,
- Properties: F_{ty}, F_{tu} = yield and ultimate strength, respectively,
 D = ductility,
 E = Young's modulus ($E = d\sigma/d\epsilon$ if $|\sigma| \leq F_{ty}$),
 E_{pl} = tangent modulus ($E_{pl} = d\sigma/d\epsilon$ if $|\sigma| > F_{ty}$),
 $\Gamma_{HCF}(\sigma_{alt})$ = stress vs. life curve for high cycle fatigue,
 $\Gamma_{LCF}(\Delta\epsilon)$ = strain vs. life curve for low cycle fatigue,
 $\Theta_{creep}(\sigma)$ = stress vs. time curve for creep rupture.

(B) Global Structural Analysis:

The global structural analysis provides the following values:

- σ_j^{oper} = steady state mean stress during operation for stress component j ,
 σ_j^{rest} = stress at rest (e.g. stress due to assembly loads) for stress component j ,
 f_{exp}^c = combined expected frequency,

- $\sigma_{j,n}^{ss}$ = sinusoidal amplitude of the stress component j for the n^{th} exciting frequency, $1 \leq n \leq n_{ss}$,
 n_{ss} = number of exciting frequencies (sinusoidal amplitudes),
 σ_j^r = root-sum-square of the random stress component j (a Rayleigh distribution is assumed),
 t_{total} = total operating time per flight.

The global response is assumed to be elastic so that Neuber's rule can be applied at a notch root.

(C) Stress History:

The continuous probability density function of the sinusoidal plus random loads in the interval⁷ $0 \leq \lambda \leq \infty$, as shown by the dotted line in figure 3, is replaced by several discrete amplitude levels $(\sigma_j^{el,nom})_{alt}^i$ and the corresponding numbers of cycles $n^{(i)}$ where $1 \leq i \leq n_{HCF}$. The bars in figure 3 show the discrete probability density function approximating the continuous probability density function. The cumulative number of cycles within the interval $\lambda_i \leq \lambda \leq \lambda_{i+1}$ is

$$n^{(i)} = \left[e^{-\frac{1}{2}\lambda_i^2} - e^{-\frac{1}{2}\lambda_{i+1}^2} \right] f_{exp}^c t_{total}.$$

⁷see equation 3 on page 4 for the meaning of λ .

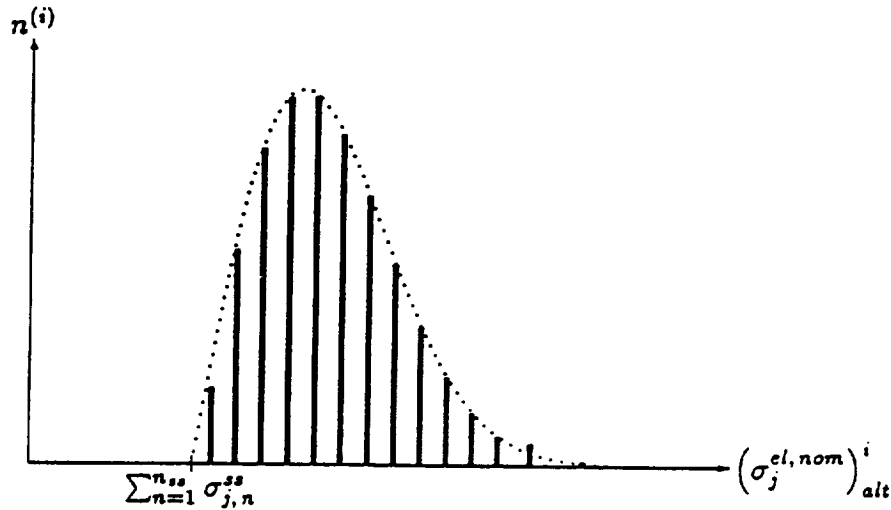


Figure 3: Comparison of the continuous and the discrete probability density function for the sinusoidal plus Rayleigh distributed random loads.

And the equivalent discrete amplitude is

$$(\sigma_j^{el, nom})_{alt}^i = \lambda_{middle} \sigma_j^r + \sum_{n=1}^{n_{ss}} \sigma_{j,n}^{ss}$$

where

$$\lambda_{middle} = \sqrt{-2 \ln \left[\frac{1}{2} \left(e^{-\frac{1}{2} \lambda_i^2} + e^{-\frac{1}{2} \lambda_{i+1}^2} \right) \right]}$$

λ_{middle} is placed in the interval $\lambda_i \leq \lambda \leq \lambda_{i+1}$ so that half of the number of cycles $n^{(i)}$ lie in the interval $\lambda_i \leq \lambda \leq \lambda_{middle}$ and half of them in $\lambda_{middle} \leq \lambda \leq \lambda_{i+1}$. Other choices for defining λ_{middle} are also frequently used.

The interval limits λ_i and λ_{i+1} can be chosen arbitrarily as long as the divisions are small enough to represent the continuous probability density function. The following limits are given as a suggestion:

$0 \leq \lambda \leq .15$	\rightarrow	in 1 step	$\Delta\lambda = .15$
$.15 \leq \lambda \leq 3.05$	\rightarrow	in 29 steps	$\Delta\lambda = .10$
$3.05 \leq \lambda \leq 3.5$	\rightarrow	in 1 step	$\Delta\lambda = .45$
$3.5 \leq \lambda \leq 5.0$	\rightarrow	in 3 steps	$\Delta\lambda = .50$
$5.0 \leq \lambda \leq \infty$	\rightarrow	in 1 step	$\Delta\lambda = \infty$

(D) Cycle Counting:

HCF	cycles: $n^{(i)}$ alt. stress: $(\sigma_{eff, K_f}^{el})_{alt}^{(i)}$ mean stress: $(\sigma_{eff}^{el})_{mean}$
LCF	cycles: 1 per flight strain range: $\Delta\epsilon_{eff}^{el-pl}$ mean strain: $(\epsilon_{eff}^{el-pl})_{mean}$
Creep	time: t_{total} per flight peak stress: $(\Delta\sigma_{eff}^{el-pl})_{first}$

(E) Notch Analysis:

The strain range for low cycle fatigue is

$$\Delta\sigma_j^{el, nom} = \sigma_j^{oper} - \sigma_j^{rest} + (\sigma_j^{el, nom})_{alt}^{(n_{HCF})}$$

$$\Delta\sigma_{eff}^{el} = \Psi(K_{t,j} \Delta\sigma_j^{el,nom})$$

$$\Delta\sigma_{eff}^{el-pl} = \begin{cases} \left(1 - \frac{E_{pl}}{E}\right) F_{ty} \left[1 + \sqrt{1 + \frac{E/E_{pl}}{(E/E_{pl}-1)^2} \left(\frac{\Delta\sigma_{eff}^{el}}{F_{ty}}\right)^2}\right] & \text{if } \Delta\sigma_{eff}^{el} > 2F_{ty} \\ \Delta\sigma_{eff}^{el} & \text{if } \Delta\sigma_{eff}^{el} \leq 2F_{ty} \end{cases}$$

$$\Delta\epsilon_{eff}^{el-pl} = \frac{(\Delta\sigma_{eff}^{el})^2}{E\Delta\sigma_{eff}^{el-pl}}$$

and the mean strain for low cycle fatigue is

$$\sigma_j^{el,nom} = \sigma_j^{oper} + (\sigma_j^{el,nom})_{alt}^{(n_{HCF})}$$

$$\sigma_{eff}^{el} = \Psi(K_{t,j} \sigma_j^{el,nom})$$

$$(\sigma_{eff}^{el-pl})_{first} = \begin{cases} \left(1 - \frac{E_{pl}}{E}\right) F_{ty} \left[\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{E/E_{pl}}{(E/E_{pl}-1)^2} \left(\frac{\sigma_{eff}^{el}}{F_{ty}}\right)^2}\right] & \text{if } \sigma_{eff}^{el} > F_{ty} \\ \sigma_{eff}^{el} & \text{if } \sigma_{eff}^{el} \leq F_{ty} \end{cases}$$

$$(\epsilon_{eff}^{el-pl})_{mean} = \frac{(\sigma_{eff}^{el})^2}{E(\sigma_{eff}^{el-pl})_{first}} - \frac{1}{2} \Delta\epsilon_{eff}^{el-pl}$$

The alternating and mean stresses for high cycle fatigue are

$$(\sigma_{eff,K_f}^{el})_{alt}^{(i)} = \Psi(K_{f,j} (\sigma_j^{el,nom})_{alt}^{(i)})$$

$$(\sigma_{eff}^{el})_{mean} = (\sigma_{eff}^{el-pl})_{first} - \Psi(K_{t,j} (\sigma_j^{el,nom})_{alt}^{(n_{HCF})})$$

The mean stress is constant for any i .

(F) Damage Accumulation:

$$(\sigma'_{mean})^{(i)} = \begin{cases} 0 & \text{if } (\sigma_{eff,K_f}^{el})_{alt}^{(i)} > F_{ty} \\ (\sigma_{eff}^{el})_{mean} & \text{otherwise} \end{cases}$$

$$\sigma_{eq,alt}^{(i)} = \frac{(\sigma_{eff,K_f}^{el})_{alt}^{(i)}}{1 - \frac{(\sigma'_{mean})^{(i)}}{F_{tu}}}$$

$$\Delta \epsilon_{eq} = \frac{1}{E} \left[\Delta \sigma_{eff}^{el-pl} + \frac{E \Delta \epsilon_{eff}^{el-pl} - \Delta \sigma_{eff}^{el-pl}}{\left(1 - \frac{(\epsilon_{eff}^{el-pl})_{mean}}{D}\right)^{.6}} \right]$$

$$\Phi_{flight} = \sum_{i=1}^{n_{HCF}} \left[\frac{n^{(i)}}{\Gamma_{HCF}(\sigma_{eqalt}^{(i)})} \right] + \frac{1}{\Gamma_{LCF}(\Delta \epsilon_{eq})} + \frac{t_{total}}{\Theta_{creep} \left((\sigma_{eff}^{el-pl})_{first} \right)}$$

For the creep damage it is conservatively assumed that the peak stress $(\sigma_{eff}^{el-pl})_{first}$ acts during the full operating time t_{total} . The factors of safety were omitted in all of the above equations.

(G) Fatigue Life:

$$N_{flights} = \frac{1}{\Phi_{flight}}$$

The assumption is made that the load history of each flight generates the same stress-strain response. This is not always the case. For example, the very first stress cycle starting from $\sigma = \epsilon = 0$ has a different stress-strain path than subsequent but otherwise identical stress cycles since the subsequent cycles do not in general start at $\sigma = \epsilon = 0$.

6.2 Example II: Total Fatigue Curve

This example uses a more refined method. The stress history is given directly as elastic stress vs. time and the damage is calculated based on the total fatigue curve. A Ramberg-Osgood stress-strain relation is used.

The function $\Psi(\sigma_j)$ again defines the effective stress based on the component stresses as given on page 16.

(A) Component Geometry and Material Properties:

The following parameters are given:

- | | |
|-------------|--|
| Geometry: | $K_{t,j}$ = elastic stress concentration factor acting on stress component j , |
| | $K_{f,j} = K_{t,j}$ — crack initiation is only considered. |
| Properties: | F_{tu} = ultimate strength, |
| | D = ductility, |
| | E = Young's modulus, |
| | $\sigma_f, \epsilon_f, b, c$ = curve fitting parameters — the fatigue curve and the cyclic stabilized stress-strain curve have to be fitted simultaneously and the fatigue curve should consider crack initiation only to be consistent with $K_{t,j} = K_{f,j}$, |
| | $\Theta_{creep}(\sigma)$ = stress vs. time curve for creep rupture. |

(B) Global Structural Analysis:

The global structural analysis provides the following values:

- $\sigma_j^{el}(r_i)$ = elastic stress component j at the stress reversal points r_i where $1 \leq i \leq n_{tot}$. $\sigma_j^{el}(r_i)$ accounts for all loads (e.g. also mean operating loads) and encompasses the whole stress history including any start/stop cycles,
- t_{total} = total operating time per flight.

The global response is again assumed to be elastic so that Neuber's rule can be applied at a notch root.

(C) Stress History:

Reorder the reversal points in the stress history so that the very first reversal point r_1 corresponds to the largest effective stress level considering the stress concentration factor

$$\Psi(K_{t,j} \sigma_j^{el}(r_1)) = \max_i [\Psi(K_{t,j} \sigma_j^{el}(r_i))].$$

(D) Cycle Counting:

The cycle counting routine is the controlling routine of a damage accumulation calculation. It identifies each strain cycle $\Delta \epsilon_{eff}^{el-pl}$ together with its mean stress σ_{mean} and

its mean strain ϵ_{mean} . Several cycle counting methods are in use in practice but the preferred one at Rocketdyne is the 'Loop Closure Method'. A verbal description of this method is given in reference [2] and a short algorithm is given below. The stress $(\sigma_{eff}^{el-pl})_{first}$ and the strain $(\epsilon_{eff}^{el-pl})_{first}$ as well as the two functions $NEUBER_{\sigma}(\Delta\sigma_{eff}^{el})$ and $NEUBER_{\epsilon}(\Delta\sigma_{eff}^{el})$ are described in the next section '(E) Notch Analysis' while the damage calculation is given in section '(F) Damage Accumulation'. The variables r_i , r_a , r_b , r_c , and r_d are pointers to several arrays storing information at the reversal points of the stress history.

• Initialize:

all $r_i = \text{'available'}$; $0 \leq i \leq n_{tot}$

$r_a = 0$; $r_b = 0$; $r_c = 0$; $r_d = 1$; $sign = +1$

$\sigma_{start}(r_1) = (\sigma_{eff}^{el-pl})_{first}$

$\epsilon_{start}(r_1) = (\epsilon_{eff}^{el-pl})_{first}$

WHILE $r_d < n_{tot}$ DO

IF $r_b = 0$ THEN

$r_a \leftarrow r_b$; $r_b \leftarrow r_c$; $r_c \leftarrow r_d$; $r_d \leftarrow r_d + 1$; $sign \leftarrow -sign$

ELSE

• Calculate stress ranges:

$$\Delta\sigma_{old}^{el} = \Psi(K_{t,j} [\sigma_j^{el}(r_b) - \sigma_j^{el}(r_a)])$$

$$\Delta\sigma_{prev}^{el} = \Psi(K_{t,j} [\sigma_j^{el}(r_c) - \sigma_j^{el}(r_b)])$$

$$\Delta\sigma_{current}^{el} = \Psi(K_{t,j} [\sigma_j^{el}(r_d) - \sigma_j^{el}(r_c)])$$

IF $r_b \neq 1$ THEN

• Calculate starting point of $\Delta\sigma_{prev}^{el}$:

$$\sigma_{start}(r_b) = \sigma_{start}(r_a) + sign \times NEUBER_{\sigma}(\Delta\sigma_{old}^{el})$$

$$\epsilon_{start}(r_b) = \epsilon_{start}(r_a) + sign \times NEUBER_{\epsilon}(\Delta\sigma_{old}^{el})$$

END

IF $\Delta\sigma_{current}^{el} > \Delta\sigma_{prev}^{el}$ THEN

• Accumulate damage with:

$$\Delta\sigma_{eff}^{el-pl} = NEUBER_{\sigma}(\Delta\sigma_{prev}^{el})$$

$$\Delta\epsilon_{eff}^{el-pl} = NEUBER_{\epsilon}(\Delta\sigma_{prev}^{el})$$

$$\sigma_{mean} = \sigma_{start}(r_b) - sign \times \Delta\sigma_{eff}^{el-pl} / 2$$

$$\epsilon_{mean} = \epsilon_{start}(r_b) - sign \times \Delta\epsilon_{eff}^{el-pl} / 2$$

• Mark reversal points r_b and r_c as 'unavailable'.

• Look for the next three largest reversal points r_a , r_b , and r_c so that

$$0 \leq r_c < r_d; \quad 0 \leq r_b < r_c; \quad 0 \leq r_a < r_b.$$

END

END
 END
 • Calculate stress range:
 $\Delta\sigma_{prev}^{el} = \Psi \left(K_{t,j} \left[\sigma_j^{el}(r_c) - \sigma_j^{el}(r_b) \right] \right)$
 • Accumulate damage with:
 $\Delta\sigma_{eff}^{el-pl} = \text{NEUBER}_\sigma \left(\Delta\sigma_{prev}^{el} \right)$
 $\Delta\epsilon_{eff}^{el-pl} = \text{NEUBER}_\epsilon \left(\Delta\sigma_{prev}^{el} \right)$
 $\sigma_{mean} = \sigma_{start}(r_b) - \text{sign} \times \Delta\sigma_{eff}^{el-pl} / 2$
 $\epsilon_{mean} = \epsilon_{start}(r_b) - \text{sign} \times \Delta\epsilon_{eff}^{el-pl} / 2$
 STOP

(E) Notch Analysis:

The very first elastic stress excursion from zero load to the peak load during operation is

$$\left(\sigma_{eff}^{el} \right)_{first} = \Psi \left(K_{t,j} \sigma_j^{el}(1) \right).$$

Solve for $\left(\sigma_{eff}^{el-pl} \right)_{first}$ from

$$\frac{\left(\sigma_{eff}^{el} \right)_{first}^2}{E} = \left(\sigma_{eff}^{el-pl} \right)_{first} \left\{ \frac{\left(\sigma_{eff}^{el-pl} \right)_{first}}{E} + \epsilon_f \left[\frac{\left(\sigma_{eff}^{el-pl} \right)_{first}}{\sigma_f} \right]^{c/b} \right\}$$

and the elastic-plastic strain becomes

$$\left(\epsilon_{eff}^{el-pl} \right)_{first} = \frac{\left(\sigma_{eff}^{el} \right)_{first}^2}{E \left(\sigma_{eff}^{el-pl} \right)_{first}}.$$

The function $\text{NEUBER}_\sigma \left(\Delta\sigma_{eff}^{el} \right)$ used in the cycle counting routine returns the solution $\Delta\sigma_{eff}^{el-pl}$ of the following equation

$$\frac{\left(\Delta\sigma_{eff}^{el} \right)^2}{E} = 2\Delta\sigma_{eff}^{el-pl} \left\{ \frac{\Delta\sigma_{eff}^{el-pl}}{2E} + \epsilon_f \left[\frac{\Delta\sigma_{eff}^{el-pl}}{2\sigma_f} \right]^{c/b} \right\}$$

and $\text{NEUBER}_\epsilon \left(\Delta\sigma_{eff}^{el} \right)$ provides the elastic-plastic strain range

$$\Delta\epsilon_{eff}^{el-pl} = \frac{\left(\Delta\sigma_{eff}^{el} \right)^2}{E \Delta\sigma_{eff}^{el-pl}}.$$

(F) Damage Accumulation:

The damage increment due to one full cycle (two reversals) is

$$\Delta \Phi_{fatigue} = 2 \left[\frac{\sigma_f}{E \epsilon_f} \left(\frac{1 - \sigma_{mean}/F_{tu}}{1 - |\epsilon_{mean}|/D} \right) \frac{\Delta \epsilon_{pl}}{\Delta \epsilon_{el}} \right]^{\frac{1}{1-c}}$$

where

$$\Delta \epsilon_{el} = \frac{\Delta \sigma_{eff}^{el-pl}}{E}$$

$$\Delta \epsilon_{pl} = \Delta \epsilon_{eff}^{el-pl} - \Delta \epsilon_{el}.$$

The damage increment is added to the previous damage

$$\Phi_{fatigue} \leftarrow \Phi_{fatigue} + \Delta \Phi_{fatigue}.$$

(G) Fatigue Life:

The total damage per flight is

$$\Phi_{flight} = \Phi_{fatigue} + \frac{t_{total}}{\Theta_{creep} \left(\left(\sigma_{eff}^{el-pl} \right)_{first} \right)}$$

and, therefore, the allowable number of flights

$$N_{flights} = \frac{1}{\Phi_{flight}}.$$

As in Example I it is again assumed that the load history of each flight generates the same stress-strain response.

7 Random Variables for the Damage Calculation

7.1 General

An attempt is made in this section to identify the parameters which can be considered random variables in a probabilistic fatigue and creep damage analysis. The choice of random variables and their interdependency is, of course, influenced by the desired sophistication of the model, the structural problem to be analyzed, and the available data to characterize the statistical distribution of the input variables. For example, simplifications or approximations are necessary if not sufficient material test data are available to fully describe the fatigue curve (S-N curve). The set of random variables given in the following sections should be considered as one possibility.

Material behavior is in general characterized by a few material properties which are often considered to be random variables. The mean value and other distribution parameters of these material properties depend on temperature among other things. It is not feasible to fully characterize the whole distribution of the material behavior at each operating temperature. Therefore, an interpolation scheme has to be devised to account for the temperature (and other) dependency so that not too many material test results are required.

The following sections only discuss random variables used for the damage calculation. The whole life prediction analysis depends on more random variables than just the ones occurring in the damage calculation. The global structural response as well as the local notch response depends for example on the geometry, material properties, and applied load history.

Due to the limited scope of this write-up it is impossible to discuss all issues relating to a probabilistic fatigue analysis. It is rather attempted to summarize some of the more important issues. A more detailed discussion is given, for example, in reference [7].

7.2 'Conventional' Method

This section suggests random variables for the 'conventional' method as discussed in section 5.1 (page 11).

- Total Damage Φ_{tot} :

In a deterministic damage analysis the total damage Φ_{tot} in equation 14 (page 11) is commonly assumed to equal 1 at failure. However, experimentally obtained values for Φ_{tot} can vary and, therefore, the total damage Φ_{tot} should be treated as a

random variable. The variation is primarily due to load interaction effects but some variability is also due to variations in the basic material behavior. The material variations should be eliminated as good as possible or somehow accounted for to determine the statistics of Φ_{tot} .

- High Cycle Fatigue Curve $\Gamma_{HCF}(\sigma_{alt})$:

The high cycle fatigue curve is commonly defined by the equation

$$N_f = C(\sigma_{alt})^m \quad (28)$$

where C and m are material properties. If enough material test data is available than both parameters C and m can be defined as correlated random variables. However, most commonly m is considered to remain constant and only C is treated as an independent random variable. The lognormal and the two parameter Weibull distributions are most often used to describe C .

If the fatigue damage is to be calculated for a variable stress history then the failure trajectory has to be defined. The failure trajectory defines the failure curve of one fatigue specimen while the median S-N curve defines the failure curve of several specimens. This subject is discussed in reference [7, pages 4-3 to 4-5]. The failure trajectories are typically assumed to be parallel to the median S-N curve.

More sophisticated models are occasionally used in practice. One such model accounts for the frequently encountered fact that the scatter in cycles to failure N_f increases with decreasing stress amplitude. This method is also discussed in reference [7, pages 5-1 to 5-8].

- Low Cycle Fatigue Curve $\Gamma_{LCF}(\Delta\epsilon)$:

The low cycle fatigue curve is commonly defined by an equation similar to equation 28

$$N_f = C(\Delta\epsilon)^m \quad (29)$$

where C and m are material properties (of course, C and m used here are different than the C and m used in equation 28).

Similar remarks as given above apply to this set of data.

- Creep Rupture Curve $t_r(\sigma)$:

In Eq. (23) the operating temperature T and stress σ may be considered as random variables. The regression constant c and Temperature T_a are determined by test results, which may be considered as randomly distributed also. For the value of c in Eq. (23) a value of $c = -20$ was initially proposed [9], but optimized values between -10 and -40 have subsequently been found to be suitable depending on the material [12].

- Mean Stress Effect:

In a deterministic analysis the modified Goodman rule as in equation 16 (page 12) is used to account for the mean stress effect if no test data is available. This bilinear rule is not always confirmed by experimentally obtained results. Besides of treating the ultimate tensile strength F_{tu} as a random variable another random variable n can be introduced

$$\sigma_{eq\ alt} = \frac{\sigma_{alt}}{\left(1 - \frac{\sigma'_{mean}}{F_{tu}}\right)^n} \quad (30)$$

- Mean Strain Effect:

See equation 21 (page 13). The exponent .6 can be treated as a random variable similar to the procedure used above.

7.3 Total Fatigue Curve

This section suggests random variables for the total fatigue curve method as discussed in section 5.2 (page 14).

- Total Damage Φ_{tot} :

See 7.2 for a discussion on this issue.

- Material Properties σ_f , ϵ_f , b , c :

Some items:

- Often b and c considered to be constant and only σ_f and ϵ_f are treated as random variables.
- If sufficient data is not available to fully characterize the distribution of σ_f then the distribution can be assumed to be similar to the distribution of the true ultimate tensile strength.
- Similar statement about ϵ_f and the true strain to failure (ductility).
- Manson's Method of Universal Slopes equation 19 (page 13) is very similar to Morrow's equation 25 (page 15). If some parameters are not known in Morrow's equation then they could be 'borrowed' from Manson's Method of Universal Slopes equation.
- Note that the parameters in equations 25 and 26 (15) are the same. Therefore, in a statistical analysis both equations have to be simultaneously fitted to the test data.

- **Mean Stress Effect:**

The mean stress effect in equation 25 (page 15) is accounted for through the factor

$$\left(1 - \frac{\sigma_{mean}}{F_{tu}}\right).$$

A similar modification as mentioned in section 7.2 can be made here. Therefore, above equation becomes

$$\left(1 - \frac{\sigma_{mean}}{F_{tu}}\right)^n$$

and n is treated as a random variable. Of course, F_{tu} is treated as a random variable anyway.

- **Mean Strain Effect:**

As above. The factor

$$\left(1 - \frac{|\epsilon_{mean}|}{D}\right)$$

becomes

$$\left(1 - \frac{|\epsilon_{mean}|}{D}\right)^n$$

where n and D are treated as random variables. The n mentioned here is *not* related to the n mentioned in the previous item!

8 Remarks to Analysis Procedures

Several limitations and restrictions to the presented analysis procedures have already been mentioned in the proceeding text. The following few points are being highlighted because of their importance.

1. **Thermomechanical Fatigue:** Strictly speaking the damage calculation presented here is applicable to isothermal loading only. However, it is a common practice at Rocketdyne to use the presented method together with the worst material properties within the applied temperature range.
2. **Neuber's Rule:** It is applicable only to situations where the notch is loaded in load control and the net section remains elastic. For displacement controlled problems Neuber's rule gives conservative answers (the predicted elastic-plastic strains are too large).
3. **Thermal Gradient:** Strictly speaking the elastic-plastic surface strain can *not* be calculated from the fictitious elastic surface stress with Neuber's rule if the stress is due to a thermal gradient. However, conservative answers can be expected if Neuber's rule is applied anyway.
4. **Multiaxial Fatigue:** The stress vs. life or strain vs. life material curves are generally obtained from uniaxial tests. The presented methods consider a multiaxial state of stress by using the von Mises effective stress. During multiaxial fatigue several slip systems can be activated and cracks can therefore form in several planes which can not be accounted for by simply using the effective stress. A more refined procedure for calculating the fatigue damage may be necessary if the effect of the multiaxial state of stress becomes significant. Note that at notches one stress component may dominate and a uniaxial state of stress may practically exist.
5. **Crack Propagation:** In the opinion of the author fatigue should be strictly applied to crack initiation and fracture mechanics to crack propagation. However, fatigue life test data commonly includes some crack propagation life. Therefore, all influences affecting the crack propagation (e.g. stress gradients, hold times in hydrogen etc.) should be closely simulated in the fatigue tests or otherwise accounted for.

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Appendix B.3
Stochastic Constitutive Equations

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STOCHASTIC CONSTITUTIVE EQUATIONS

Polycrystalline Austenitic Stainless Steels

0) Purpose

The purpose of this document is to suggest constitutive models which should be suitable for describing the behavior of austenitic stainless steels. Some of the models are fairly crude, and consequently, their deficiencies and limitations are pointed out where appropriate. Also, in order to suggest random constitutive models, an attempt had been made to identify which parameters of the models could be treated as being random. Finally, *this document should be considered a working document*, since with experience, better ways to approach the modeling may be found.

1) Elastic Properties

Other than possible anisotropy due to the presence of texture, microstructure has no significant effect on the elastic properties.

2) Coefficients of Thermal Expansion

Other than possible anisotropy due to the presence of texture, microstructure has no significant effect on the coefficients of thermal expansion.

3) Uniaxial Tensile Stress-Strain Behavior

From Ref [1], page 251,

$$\sigma_y^0 = 15.4 f(T) \{ 4.4 + 23(C) + 1.3(Si) + 0.24(Cr) + 0.94(Mo) + 1.2(V) + 0.29(W) + 2.6(Nb) + 1.7(Ti) + 0.82(Al) + 32(N) + 0.16(\delta \text{ ferrite}) + 0.46 d^{-0.5} \} \quad (1)$$

where

σ_y^0 = engineering yield stress in MPa with no previous cold work,

d = grain diameter in mm,

(C) = wt.-% carbon, and

(Si) = wt.-% silicon, etc.

The quantities (C), (Si), ..., (N), (δ ferrite) and d may be taken as primitive random variables.

The function $f(T)$ is included to account for the dependence of σ_y^0 on temperature T . It will be described in more detail below (see Eqn 12).

In order to develop a simple model for the stress-strain curve, it is assumed that the material obeys the power law

$$S = S_y^0 \left(\frac{e^T}{e_y^0} \right)^m, \quad e_y^0 < e^T, \quad (2)$$

where

- S = true stress,
- S_y^0 = true stress at initial yield with no previous cold work,
- e^T = total true strain,
- e_y^0 = true strain when $S = S_y^0$, and
- m = strain hardening exponent.

The strain hardening exponent m depends on temperature. This will be discussed below (see Eqn 13). The quantities S_y^0 and e_y^0 are given by

$$\epsilon_y^0 = \frac{\sigma_y^0}{E}, \quad S_y^0 = \sigma_y^0 (1 + \epsilon_y^0), \quad e_y^0 = \ln(1 + \epsilon_y^0), \quad (3)$$

where

- σ_y^0 = given by Eqn (1),
- E = Young's modulus (which depends on temperature), and
- ϵ_y^0 = engineering strain when $S = S_y^0$.

It should be noted that Eqn (2) does not provide a good representation of the stress-strain curve in cases where significant amounts of martensite form during straining. Strain-induced martensite formation can be substantial in stainless steels with low Ni content.

Eqn (2) is for solution treated material, and does not account for any cold work which may have been performed on the material prior to tension testing. To account for the cold work, let e^{P0} be the effective plastic true strain induced by previous cold working. If the working was accomplished by unidirectional rolling, then e^{P0} is given by

$$e^{P0} = -\frac{2}{\sqrt{3}} \ln(1 - RA), \quad 0 < RA < 1, \quad (4)$$

where RA is the reduction in area, which could be considered to be a primitive random variable. The stress-strain curve for the pre-cold worked material is then

$$S = S_0^0 \left(\frac{e + e^{P0}}{e_0^0} \right)^m, \quad (5)$$

where

e = total true strain as measured from the beginning of the tension test.

The engineering yield stress σ_y depends on the amount of previous cold work. The value of σ_y for the worked material may be computed implicitly from

$$\sigma_y \lambda = S_0^0 \left(\frac{e^{P0} + \ln \lambda}{e_0^0} \right)^m, \quad \lambda = 1 + \frac{\sigma_y}{E}. \quad (6)$$

Given σ_y , one can compute

$$\epsilon_y = \frac{\sigma_y}{E}, \quad S_y = \sigma_y (1 + \epsilon_y), \quad e_y = \ln(1 + \epsilon_y), \quad (7)$$

where

σ_y = value of σ at the initial yield point of the tension test,
 σ = engineering stress as measured from the beginning of the tension test,
 ϵ_y = value of ϵ at $\sigma = \sigma_y$,
 ϵ = engineering strain as measured from the beginning of the tension test,
 S_y = value of S at $\sigma = \sigma_y$, and
 e_y = value of e at $\sigma = \sigma_y$.

Now, at maximum load,

$$e_m = m - e^{P0}, \quad S_m = S_0^0 \left(\frac{m}{e_0^0} \right)^m, \quad (8)$$

where

e_* = value of e at maximum load, and

S_* = value of S at maximum load.

The engineering ultimate tensile strength σ_* is

$$\epsilon_* = \exp(e_*) - 1, \quad \sigma_* = \frac{S_*}{1 + \epsilon_*}, \quad (9)$$

where

ϵ_* = value of ϵ at maximum load, and

σ_* = value of σ at maximum load.

Note that ϵ_* , σ_* , e_* , S_* , ϵ_* , σ_* and e_* all depend on the amount of previous cold work, while S_* does not.

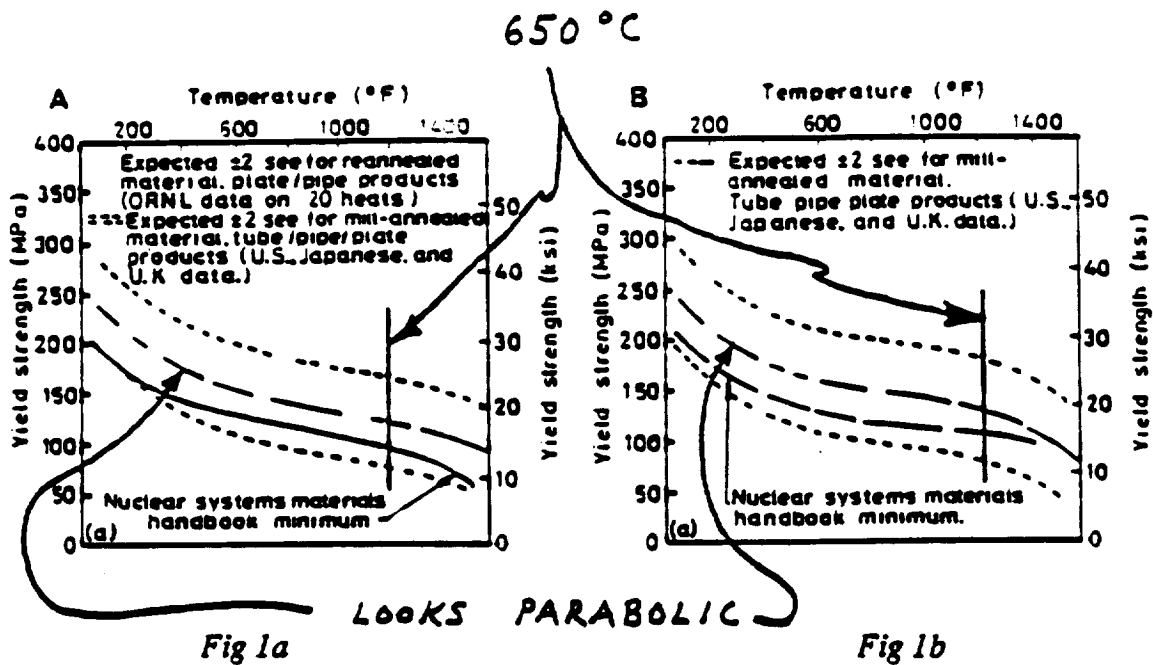
Now, the hardening curve should be specified to NESSUS in terms of S vs. e^P for finite deformation analyses. Thus, given e and S from Eqn (5),

$$e^P = e - \ln \left(1 + \frac{S}{E \exp(e)} \right). \quad (10)$$

For small strain analyses, the hardening curve should be specified in terms of σ vs. ϵ^P , where

$$\epsilon^P = \exp(e^P) - 1, \quad \sigma = \frac{S}{\exp(e)}. \quad (11)$$

As mentioned above, the yield stress depends on temperature. Fig 1a directly below shows the dependence of yield stress on temperature for type 304 stainless; Fig 1b, for type 316 (these figures are from Ref [2], page 88).



While these curves are for steels in a cold worked condition, Figs 1 nevertheless suggest that, for room temperature $< T < \sim 650^{\circ}\text{C}$, the yield stress decreases with increasing temperature in a parabolic fashion. Thus, $f(T)$ in Eqn (1) should look like

$$f(T) = k_0 + k_1 T + k_2 T^2 \quad , \quad (12)$$

where $f(T) = 1$ at room temperature, and k_0 , k_1 and k_2 are experimentally determined. When determining k_0 , k_1 and k_2 from tension tests at various temperatures, one should ensure that all of the tests are performed at the same (reference) strain rate. This is because the value of σ_y^0 is sensitive to strain rate at higher temperatures.

Young's modulus E also depends on temperature. As a first step in constructing E as a function of temperature, Table 1 directly below may be of use, where E is in GPa (data taken from Ref [3], page 13; and Ref [4], page 594).

	25°C	343°C	436°C
AISI 304	193	149	130
AISI 316	193	152	127
AISI 348	–	150	132

Table 1

As temperature increases, m also generally decreases. At room temperature, $m = 0.50$ for AISI 300 series standard austenitic stainless steels 301, 302, 304, 310, 316, 321 and 347 (see Table 11 in Ref [1]). From page 100 of Ref [2], $m = 0.29$ at $T = 650^\circ\text{C}$ for type 304 stainless. Thus, a first approximation is

$$m = 0.5084 - 0.000336T \quad , \quad (13)$$

where T is the temperature in $^\circ\text{C}$. Note that $m = 0.50$ at $T = 25^\circ\text{C}$ in Eqn (13). (Data for the strain hardening exponent m at $T = 593^\circ\text{C}$ for types 304, 316, 321 and 348 stainless can be found on page 390 of Ref [5]). Also, T in Eqns (1), (12) and (13) could be treated as a primitive random variable.

As a final comment, the model as given by Eqns (1) through (13) is such that the ductility decreases with increasing temperature, which is consistent with experimental data in the range $25^\circ\text{C} < T < \sim 650^\circ\text{C}$. At higher temperatures though ($T > \sim 650^\circ\text{C}$), it appears the ductility begins to increase with increasing temperature, contrary to the current model (see the data on page 88 of Ref [2]).

4) Low Cycle Fatigue

A strain-life approach is taken:

$$N_f = \frac{1}{2} \left(\frac{\Delta \epsilon^p}{2 \epsilon_f} \right)^c \quad , \quad (14)$$

where

$\Delta \epsilon^p$ = plastic strain range,
 N_f = number of cycles to failure,
 ϵ_f = half-cycle fatigue ductility, and
 c = fatigue ductility exponent.

Some typical values of c (taken from Ref [3], page 13) are given in Table 2 directly below.

	343°C	436°C
AISI 304	-1.41	-1.15
AISI 316	-1.75	-1.35
AISI 348	-1.90	-1.79

Table 2

The low cycle fatigue behavior of austenitic stainless steels is quite complex, especially at higher temperatures. Apparently, the mechanisms of cracking can be different depending on temperature, frequency of cycling, and minor differences in alloy composition. Also, fatigue limits may or may not exist. Thus, Eqn (14) is probably somewhat crude.

In any case, the quantity ϵ_f is usually on the order of the true strain at fracture e_f . Fig 2 directly below (taken from Ref [1], page 256) shows e_f and e_m (e_m = true strain at maximum load) for 17 wt.-% Cr stainless as a function of Ni content at room temperature.

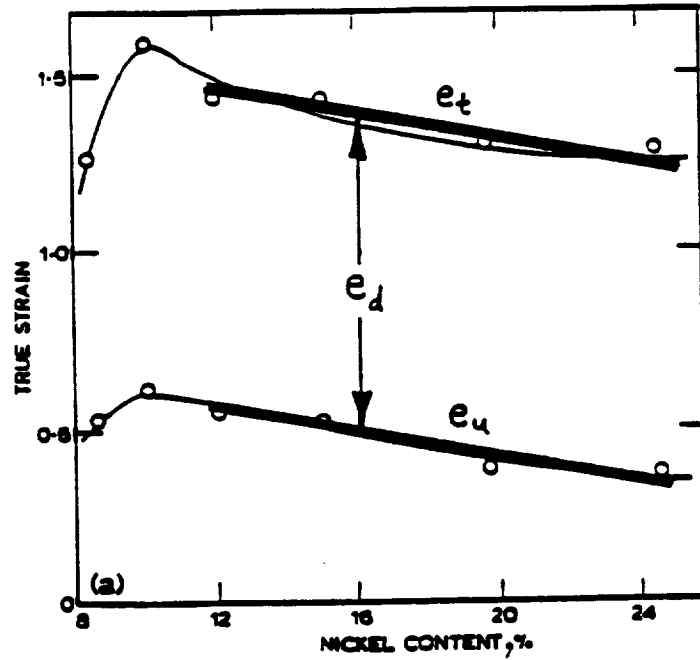


Fig 2

The non-monotonic behavior on the left-hand side of the plot is due to the formation of strain-induced martensite. For higher alloy contents though (marked by the thick, straight lines), the difference between e_t and e_u appears to be constant ($= e_d$). Thus, the relation

$$\epsilon_f = e_u + e_d \quad (15)$$

is suggested, where $e_d = 0.90$ at $T = 25^\circ\text{C}$. From page 100 of Ref [2], $e_d = 0.05$ at $T = 650^\circ\text{C}$.

So, a first approximation is

$$e_d = 0.934 - 0.00136T, \quad (16)$$

where T is the temperature in $^\circ\text{C}$. The effects of temperature and previous cold work on ϵ_f can be taken into account by calculating e_u in Eqn (15) from Eqns (4), (8) and (13).

The primitive random variables of this low cycle fatigue model are RA and T . It may also be desirable to make c random. If one chooses to do this, then c should probably exhibit more variability at higher temperatures than at lower ones (as suggested by Table 2). Also, one should be warned that this model does not take into account the effects of cycling frequency, which could become fairly pronounced at higher temperatures. A phenomenological way of accounting

for the cycling frequency is outlined in Ref [3]. Finally, it is well-known that environmental effects sometimes play an important role in low cycle fatigue behavior at elevated temperatures, and this is not accounted for here.

5) High Cycle Fatigue

A stress-life approach is taken:

$$N_f = \frac{1}{2} \left(\frac{S_e}{S_f} \right)^b, \quad (17)$$

where

- S_e = effective amplitude of stress,
- N_f = number of cycles to failure,
- S_f = half-cycle fatigue strength, and
- b = fatigue strength exponent.

Typical values of b (taken from Ref [3], page 13) are given in Table 3 directly below.

	343°C	436°C
AISI 304	-5.35	-10.64
AISI 316	-6.99	-8.55
AISI 348	-9.80	-7.63

Table 3

Mean stress effects are accounted for with the Goodman relation:

$$S_e = \frac{\sigma_a \sigma_u}{\sigma_u - \sigma_m}, \quad (18)$$

where

$\sigma_a = (1/2)(\sigma_{\max} - \sigma_{\min}) = \text{stress amplitude,}$

$\sigma_m = (1/2)(\sigma_{\max} + \sigma_{\min}) = \text{mean stress, and}$

$\sigma_u = \text{engineering ultimate tensile strength,}$

and

$\sigma_{\max} = \text{maximum stress, and}$

$\sigma_{\min} = \text{minimum stress.}$

The quantity S_f is usually on the order of the true monotonic fracture strength. Thus, a (probably) conservative approximation is

$$S_f = S_u \quad (19)$$

where

$S_u = \text{true stress corresponding to } \sigma_u.$

The effects of temperature, composition, etc., may be taken partially into account by calculating the quantities S_u and σ_u with Eqns (1) through (13) of Section 3.

Some words of caution are now in order. Eqn (17) does not take into account the effects of the cycling frequency. This may not be so bad though, since cycling frequency affects high cycle fatigue less than it does low cycle fatigue. Also, one should be careful to ensure that Eqns (17) through (19) produce the correct trends. On one hand, as temperature increases S_f tends to decrease, which tends to reduce the life. On the other hand, as seen from Table 3, b tends to increase in magnitude as temperature increases for types 304 and 316 stainless, which tends to increase the life (this presumably is because higher temperatures tend to diffuse the slip behavior). Thus, there appears to be two competing mechanisms, and one should ensure that the correct one dominates for the particular situation that happens to be under consideration.

6) Fracture Mechanics Approach

Figs 3 (taken from pages 392 and 393 of Ref [5]) show the fatigue crack propagation characteristics of types 304 and 316 stainless as a function of temperature. (Similar data for types 321

and 348 stainless may also be found in Ref [5]). The data shown in the figures were obtained by cycling at $R = 0$ and 10 cycles per minute, where $R = K_{\min}/K_{\max}$, K_{\min} = the stress intensity factor at minimum load, and K_{\max} = the stress intensity factor at maximum load.

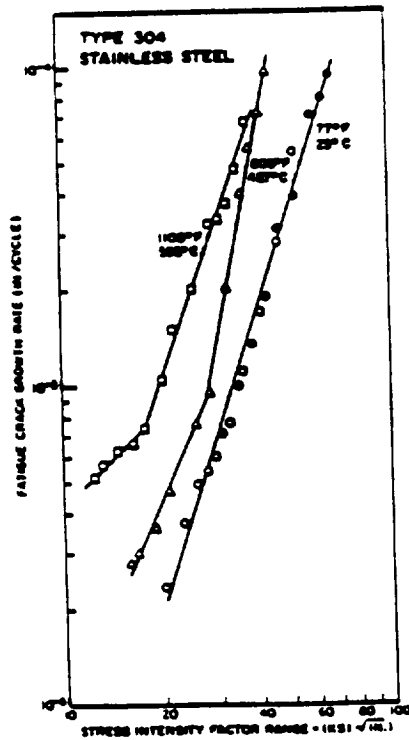


Fig 3a

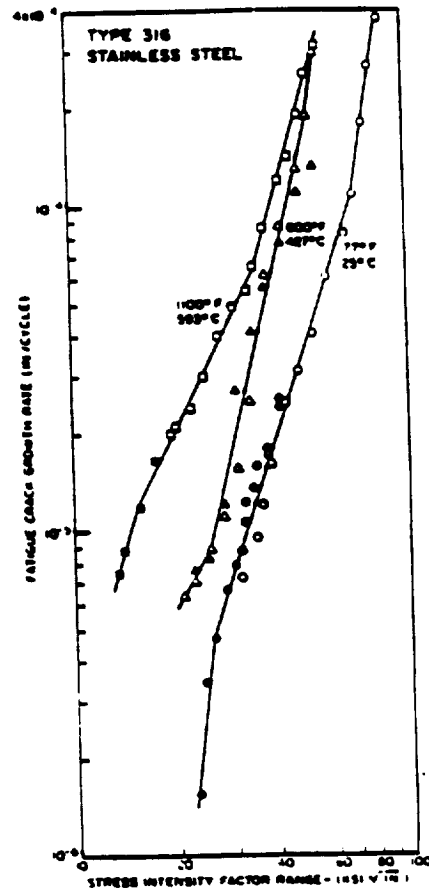


Fig 3b

As is evident by the "kinks" in the above curves of crack growth rate (\dot{a}) vs. stress intensity factor range ($\Delta K = K_{\max} - K_{\min}$), different mechanisms of growth are exhibited depending on ΔK and temperature. Also, as indicated by Fig 4 below (from page 396, Ref [5]), the behavior at room temperature is quite insensitive to alloy composition, whereas at higher temperatures, the behavior is quite sensitive to alloy composition. (The meaning of the thick, parallel dashed lines in the figure will be described below). We now look at two different methods of describing the fatigue crack propagation characteristics.

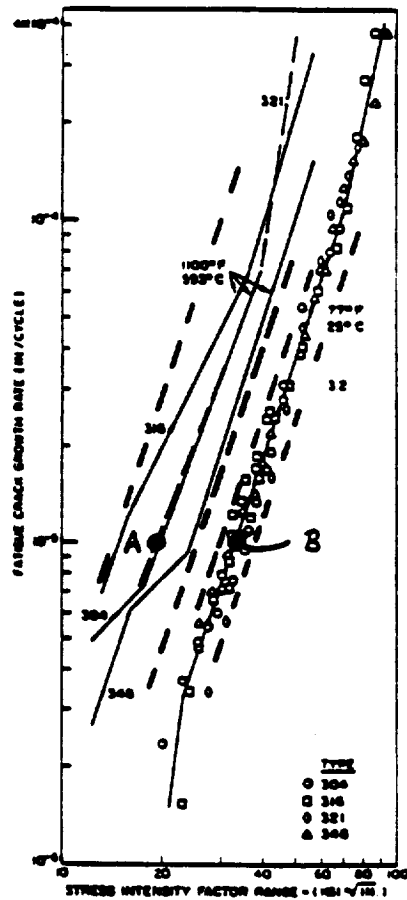


Fig 4

The first method is quite simple in that the crack growth rate is assumed to follow a single Paris-type equation:

$$\dot{a} = C (\Delta K)^n \quad , \quad (20)$$

where

- \dot{a} = crack growth rate (here, units = in/cycle),
- ΔK = stress intensity factor range (here, units = ksi $\sqrt{\text{in}}$),
- C = Paris coefficient, and
- n = Paris exponent.

In Fig 4, the width of the scatter is marked approximately by the thick, parallel dashed lines of slope $n = 3.2$. The scatter at 1100°F appears to be ~2.5 times that at 77°F. This may be expressed as

$$\frac{\sigma_C}{\sigma_C^0} = 0.8871 + 0.001466T, \quad (21)$$

where

σ_C = standard deviation of the Paris coefficient C ,

σ_C^0 = σ_C at 77°F, and

T = temperature in °F.

Using the points A and B in Fig 4 to calculate C , one obtains $\log C = -9.88$ at 77°F and $\log C = -9.12$ at 1100°F. If these are taken to be indicative of the mean values, then (approximately),

$$\log \mu_C = -9.937 + 0.0007429T, \quad (22)$$

where

μ_C = mean of the Paris coefficient C , and

T = temperature in °F.

Thus, a reasonable random fatigue crack propagation law is obtained if C in Eqn (20) is described by Eqns (21) and (22), and if in Eqn (20)

$$n = 3.2 \quad (23)$$

is used.

A couple of comments on this first method are now in order. First, T should not be random, since if it were, the statistics of C would be random. Also, besides C , the other random variables of the model would be a_i (an initial flaw size) and a_f (a final or failure flaw size).

A second method is now described and illustrated for the type 304 stainless depicted in Fig 3a. As is evident, there are two distinct regimes of behavior: a low ΔK regime and a high ΔK regime. Let the low ΔK regime be defined by

$$\dot{a}_1 = C_1 (\Delta K)^{n_1} , \quad (24)$$

and the high ΔK regime by

$$\dot{a}_2 = C_2 (\Delta K)^{n_2} , \quad (25)$$

where here, as before, \dot{a}_1 and \dot{a}_2 have units in/cycle and ΔK has units ksi $\sqrt{\text{in}}$. The values of C_1 , n_1 , C_2 and n_2 given in Table 4 below fit the data in Fig 3a quite well.

	$\log C_1$	n_1	$\log C_2$	n_2
1100°F	-6.37	1.0	-8.60	2.8
800°F	-8.19	2.2	-12.65	5.3
77°F	-9.57	3.0	-	-

Table 4

From Table 4, the following least square linear fits are obtained:

$$\log C_1 = 0.0029038 T - 9.9563 ,$$

$$\log C_2 = 0.0135 T - 23.452 ,$$

$$n_1 = -0.0017986 T + 3.252 ,$$

$$n_2 = -0.0083333 T + 11.967 , \quad (26)$$

where T is in °F. Thus, for type 304 stainless, random Paris-type behavior is achieved by letting T in Eqns (26) be random, and by calculating \dot{a} from

$$\dot{a} = \max(\dot{a}_1, \dot{a}_2) . \quad (27)$$

As before, the initial and final flaw sizes (a_i and a_f) may also be random. Similar procedures can be used to describe the fatigue crack growth characteristics of other series 300 steels.

In summary, for Paris-type behavior, ΔK is a function of the stress range $\Delta \sigma$ and the flaw size a , and the growth rate \dot{a} is a function of ΔK :

$$\Delta K = \Delta K(\Delta \sigma, a) \quad , \quad \dot{a} = \dot{a}(\Delta K) \quad . \quad (28)$$

Now, it is assumed that the major effect of microstructure is to cause the so-called "anomalous" short crack behavior. If a is less than some microstructurally determined value a_μ , then the crack growth rate is "anomalously" high. The parameter a_μ is approximately

$$a_\mu \approx 5d \quad , \quad (29)$$

where d is the grain size. Thus, to account for the short crack behavior, \dot{a} is calculated as follows: (i) if $a > a_\mu$, then the Paris-type relations (20) through (23), or (24) through (27), are used; (ii) if $a < a_\mu$, then \dot{a} is given by

$$\Delta K_\mu = \Delta K(\Delta \sigma, a_\mu) \quad , \quad \dot{a} = \dot{a}(\Delta K_\mu) \quad . \quad (30)$$

Of course d , or equivalently a_μ , may be random.

7) Creep Rupture

Describe the time to rupture with the Larson-Miller parameter (Ref [6], page 164):

$$\log t_R = \frac{P(\sigma)}{T + 273} + Y \quad , \quad (31)$$

where

t_R	= time to rupture in hours,
T	= temperature in °C,
Y	= a material property,
σ	= applied stress, and
P	= Larson-Miller parameter.

Note that Y varies somewhat from material to material, and that $Y = -18$ for 18 wt.-% Cr-8 wt.-% Ni stainless steels (AISI types 301, 302 and 304). Additionally, P is proportional to the activation energy required for creep, and thus, also conceivably varies from material to material. Note that P is an experimentally determined function of stress (P usually decreases with increasing stress).

Before discussing how Eqn (31) should depend on microstructure and composition, it should be mentioned that Chapter 5 (pages 233-353) of Ref [2] gives a fairly thorough overview, with over 400 references, of the creep behavior of austenitic stainless steels. What is evident from the overview is that the creep behavior of these materials is fairly complex. The mechanisms under which creeping occurs, and hence the overall creep behavior, can depend sensitively on grain size, temperature, applied stress, composition, previous cold working and previous ageing.

It is known that small amounts of cold work (up to ~20%) can increase creep resistance. Larger amounts of cold work, though, can actually decrease creep resistance by promoting recrystallization. Previous ageing which results in precipitation hardening also increases creep resistance, but overageing reduces it. Incorporating these effects into a simple model is difficult, and will not be attempted here (it appears that a more involved, unified approach would be required).

An approximate method to account for the effects of composition is now illustrated. From Ref [2] (page 260), for AISI 304 stainless at 650 °C:

$$\sigma_4 = 90.81 + 115(\text{Mo}) + 498.5(\text{W}) \quad , \quad (32)$$

where

σ_4 = stress (in MPa) corresponding to $t_R = 10^4$ h,

(Mo) = wt.-% molybdenum, and

(W) = wt.-% tungsten.

Formulae giving σ_4 for other temperatures, and for AISI 316, may also be found on page 260 of Ref [2]. Now, Eqn 32 (or something similar) can be substituted into Eqn (31) to construct the approximate dependence of Y on composition:

$$Y = Y(\text{composition}) = 4 - \frac{P(\sigma_4)}{923} \quad . \quad (33)$$

A word of caution is in order. Note that Eqn (32) is based on $t_R = 10^4$ h, while times of interest for the SSME are undoubtedly much less. Obviously, finding or constructing a formula like Eqn (32) based on $t_R = 10$ -100 h would be more desirable. Also, substituting Eqn (32) into

Eqn (31) to obtain Eqn (33) assumes that the activation energy for creep is independent of composition (which cannot be entirely true). Nevertheless, trends for t_R predicted by using Eqn (31) along with a relation constructed similarly to Eqn (33) should be basically (or at least qualitatively) correct.

The effect of varying temperature and applied stress is to cause different mechanisms of creep to activate and deactivate. The mechanism which dominates for a particular stress-temperature regime can be determined from "deformation mechanism maps" (such as those in Chapter 5 of Ref [2]). The effect of grain size (d) on creep resistance depends on the particular mechanism which happens to be active. For dislocation controlled (or power-law) creep, which tends to be active at higher stresses and lower temperatures, the effect of grain size is relatively weak (Ref [7]), and probably can be neglected. For diffusive creep though, the effects of grain size can be significant. In the case of Nabarro-Herring (or volumetric diffusion) creep, $t_R \propto d^2$; and in the case of Coble (or grain boundary diffusion) creep, $t_R \propto d^3$ (Ref [6], pages 148-149).

In line with the above discussion, a simple time to rupture model which incorporates the effects of temperature, composition and grain size is as follows. First, for a steel with reference composition and reference grain size (d_0), determine the function $P = P(\sigma)$ and the value of Y in Eqn (31) from experimental data (call the value of Y so determined Y_0). Next, using an equation like (32), determine a relation like (33), i.e., $Y = Y(\text{composition})$, where $Y_0 = Y(\text{reference composition})$. The time to rupture including the effects of temperature and composition is then

$$\log t_R = \frac{P(\sigma)}{T + 273} + Y(\text{composition}) \quad (34)$$

Grain size is taken into account by

$$\frac{t_R^\mu}{t_R} = \left(\frac{d}{d_0} \right)^q \quad (35)$$

where

d = grain size,
 d_0 = reference grain size,

Appendix B.4
Probabilistic Analysis of Structures Involving Random Stress-Strain Behavior

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PROBABILISTIC ANALYSIS OF STRUCTURES INVOLVING RANDOM STRESS-STRAIN BEHAVIOR

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Methodology

This methodology characterizes the uniaxial engineering stress-strain curve shown in Figure 1 by five engineering parameters. The parameters are

- Elastic modulus (E)
- Engineering stress at initial yield (σ_y)
- Initial plastic hardening slope (h_y)
- Engineering stress at point of ultimate load (σ_u)
- Engineering strain at point of ultimate load (ϵ_u)

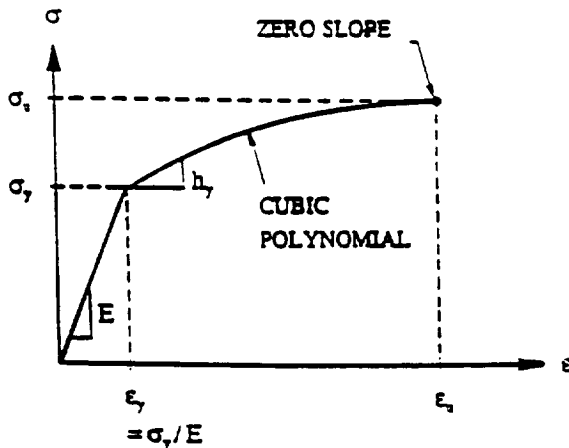


Figure 1. Uniaxial Engineering Stress-Strain Curve

Uncertainties in the stress-strain behavior of a structure can then be simulated by letting these parameters be random. Perturbations in the stress-strain curve, needed to obtain sensitivities, are then simulated by perturbations in these parameters.

The approximations are given below in equation form.

$$\begin{aligned} \sigma(\epsilon) &= E\epsilon & 0 < \epsilon < \epsilon_y & \quad (1) \\ \sigma(\epsilon) &= \sigma_y g_1(\epsilon) + \sigma_u g_2(\epsilon) + h_y g_3(\epsilon) & \epsilon_y < \epsilon < \epsilon_u \end{aligned}$$

where $\epsilon_y = \sigma_y / E$ and $g_i (i = 1, 2, 3)$ are the cubic polynomial shape functions. The boundary conditions are $\sigma(\epsilon_y) = \sigma_y$, $\sigma(\epsilon_u) = \sigma_u$, $\sigma'(\epsilon_y) = h_y$, and $\sigma'(\epsilon_u) = 0$. The last boundary condition enforces the slope at the ultimate strain to be zero. Applying these four boundary conditions determines the cubic shape functions uniquely. The boundary conditions in terms of g_i 's are:

$$\begin{aligned} g_1(\epsilon_y) &= 1 & g_1(\epsilon_u) &= 0 & g'_1(\epsilon_y) &= 0 & g'_1(\epsilon_u) &= 0 \\ g_2(\epsilon_y) &= 0 & g_2(\epsilon_u) &= 1 & g'_2(\epsilon_y) &= 0 & g'_2(\epsilon_u) &= 0 \\ g_3(\epsilon_y) &= 0 & g_3(\epsilon_u) &= 0 & g'_3(\epsilon_y) &= 1 & g'_3(\epsilon_u) &= 0 \end{aligned} \quad (2)$$

Solving the equations for the polynomial coefficients yields

$$\begin{aligned} g_1(\epsilon) &= 2z^3 - 3z^2 + 1 \\ g_2(\epsilon) &= -2z^3 + 3z^2 \\ g_3(\epsilon) &= \frac{E\epsilon_u - \sigma_y}{E} (z^3 - 2z^2 + z) \end{aligned} \quad (3)$$

$$\text{where } z = \frac{E\epsilon - \sigma_y}{E\epsilon_u - \sigma_y} \text{ varies from 0 to 1.}$$

In order to define the hardening characteristics of the material for a small strain analysis, the uniaxial curve is defined for NESSUS/FEM in terms of engineering stress σ vs. plastic engineering strain ϵ^p . For a small strain analysis, NESSUS/FEM also requires that the curve of engineering backstress ω vs. ϵ^p be entered. In this model, ω is given by the simple relation

$$\omega = \kappa(\sigma - \sigma_y) \quad (4)$$

where κ determines the type of hardening. $\kappa = 0$ indicates isotropic hardening, $\kappa = 1$ indicates kinematic hardening, and $0 < \kappa < 1$ corresponds to a mixed hardening rule.

If a finite deformation analysis is to be performed, it is more appropriate to enter the hardening description into NESSUS/FEM in terms of true stresses and strains. In this case, the uniaxial hardening curve is prescribed in terms of true stress S vs. plastic true strain ϵ^p ,

$$\begin{aligned} S &= \frac{E^3(1 + \epsilon)\sigma}{(E + \sigma)(E - \nu\sigma)^2} \\ \epsilon^p &= \log \left(\frac{E + E\epsilon}{E + \sigma} \right) \end{aligned} \quad (5)$$

In summary, the hardening model for NESSUS/FEM is defined completely by seven parameters: five from the description of the stress-strain curve, i.e., E , σ_y , h_y , σ_u , and ϵ_u ; one from the introduction of backstress, i.e., κ ; and one from the introduction of true measures for finite deformation analysis, i.e., ν , Poisson's ratio.

*Member AIAA, ASME

**Member AIAA

Restrictions on Stress-Strain Parameters

Obviously, the statistics of the seven parameters cannot be prescribed arbitrarily, since meaningless hardening curves may result. Certain restrictions must be imposed on the engineering parameters in order to ensure that a physically reasonable stress-strain model will result from the cubic polynomial. In particular, we want to ensure that the stress-strain curve is monotonically increasing, $\sigma_y < \sigma(\epsilon) < \sigma_u$ for $\epsilon_y < \epsilon < \epsilon_u$, and that it is convex upward, $\sigma''(\epsilon) < 0$ for $\epsilon_y < \epsilon < \epsilon_u$. Mathematically, this can be written as

$$\sigma''(\epsilon) = \frac{2E}{E\epsilon_u - \sigma_y} \{3(h_y - 2h_c)\chi - (2h_c - 3h_y)\} < 0 \quad (6)$$

where

$$h_c = \frac{E(\sigma_u - \sigma_y)}{E\epsilon_u - \sigma_y} \quad (7)$$

and

$$\chi = \frac{E\epsilon - \sigma_y}{E\epsilon_u - \sigma_y} \quad (8)$$

The restrictions become

$$\frac{1}{3}h_y < \frac{E(\sigma_u - \sigma_y)}{E\epsilon_u - \sigma_y} < \frac{2}{3}h_y, E > 0, \sigma_u > \sigma_y, \sigma_y > 0 \quad (9)$$

These conditions will ensure that the stress-strain curve is monotonically increasing and convex upward for $\epsilon_y < \epsilon < \epsilon_u$.

Transformation to More Natural Parameters

Because of the restrictions on the stress-strain parameters, the limits of some random variables depend on the limits of other random variables. In order to alleviate this situation, a set of new parameters, X_i 's, is developed such that the limits of these new parameters are constants. These new parameters are:

$$\begin{aligned} X_1 &= -\frac{E(\sigma_u - \sigma_y)}{E\epsilon_u - \sigma_y} + \frac{2}{3}h_y \\ X_2 &= \frac{E(\sigma_u - \sigma_y)}{E\epsilon_u - \sigma_y} - \frac{1}{3}h_y \\ X_3 &= \sigma_u - \sigma_y \\ X_4 &= \sigma_y \\ X_5 &= E \\ X_6 &= v \\ X_7 &= \kappa \end{aligned} \quad (10)$$

or inversely,

$$\begin{aligned} E &= X_5 \\ \sigma_y &= X_4 \\ h_y &= 3(X_1 + X_2) \\ \sigma_u &= X_3 + X_4 \\ \epsilon_u &= \frac{X_3}{X_1 + 2X_2} + \frac{X_4}{X_5} \\ v &= X_6 \\ \kappa &= X_7 \end{aligned} \quad (11)$$

The two new parameters, v and κ , are necessary to complete the description of the stress-strain curve to NESSUS and can also be random. Thus, the original engineering parameters are transformed into more natural parameters, X_i 's. The relationship and its inverse, although not linear, is easy to determine.

The restrictions on the X_i 's are now simple:

$$\begin{aligned} X_i &> 0 \quad (i = 1, 2, 3, 4, 5) \\ -1 &\leq X_6 \leq \frac{1}{2} \\ 0 &\leq X_7 \leq 1 \end{aligned} \quad (12)$$

Correlation Among Parameters

The current version of NESSUS requires statistically independent random variables. However, in general the engineering parameters and $E, \sigma_y, \sigma_u, h_y, \epsilon_u, v$, and κ or X_i 's will most likely be correlated. Thus, a general procedure to handle correlated nonnormal variables has been developed. The correlated variables X_i 's are transformed to independent, uncorrelated variables Z_i 's. The procedure is based on the Nataf transformation [9,10] and is an extension of the method discussed in [6]. This method constructs a joint probability density function for the X_i 's which satisfies their marginal distributions and correlation coefficients. The Nataf transformation is valid for any continuous marginal distribution although the initial implementation has been set up for lognormal, normal, Weibull, Frechet, and Extreme Value distributions only. The random field problem, i.e., the random stress-strain parameters varying spatially over the structure, can be handled with this method, but, it is not presently implemented in the code.

Summary of Transformation Procedure

- Compute statistics for the X_i 's. These can be obtained directly from the experimental results of stress-strain curves.
- Transform the correlated X_i 's to independent uncorrelated variables Z_i 's.

Solution Procedure

The solution procedures already present in NESSUS can be used to analyze structures with random stress-strain response. Thus, the probabilistic analysis capabilities already developed can be utilized. The steps necessary to compute the probabilistic response of a structure are outlined below.

- Define the structure to NESSUS by constructing a finite element model.
- Define which quantities are the random variables along with the statistical information, mean, standard deviation, and distribution type for each random variable. The distribution type must be continuous and can be chosen as: uniform, normal, lognormal, Weibull, extreme value, maximum entropy, Frechet, truncated normal, and truncated Weibull.
- Select the response to analyze, i.e., displacement, frequency, buckling, fatigue, etc. The response may involve coupling the finite element results with resistance models[11]. If so, a predefined response from a NESSUS library may be selected or the user may program his own.
- Select the probabilistic method options and the cumulative distribution output points.

Specification of Stress-Strain Curve to NESSUS

The specification of the stress-strain curve to NESSUS can be done easily in the user-written subroutine UWKSL. This subroutine has on input the equivalent plastic strain value, the temperature at the node, the node number and the X_i values. The equations which parameterize the stress-strain curve in terms of a cubic polynomial are then used to compute the yield stress at this plastic strain, the slope of the work hardening curve, and the backstress value. The NESSUS probabilistic algorithms perturb the X_i 's in order to compute sensitivities and pass the perturbed values to UWKSL. The perturbed stress-strain curve is computed using the same equations as before. A representative UWKSL routine is given in the appendix.

Example Problem

An example problem is analyzed to demonstrate the capabilities of the code. The problem analyzed is that of a thick cylinder under internal pressure, Figure 2. The internal pressure and the stress-strain curve are random.

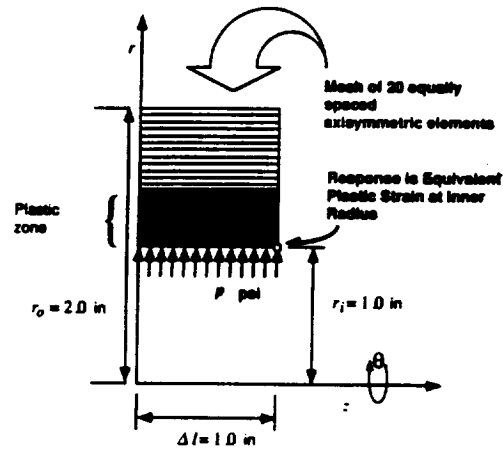


Figure 2. Thick-walled Cylinder Under Internal Pressure

The stress-strain parameters are represented by lognormal distributions. In general, these parameters are correlated; however, for this example they are assumed independent. The cumulative distribution function of the equivalent plastic strain at the inner radius is computed. In addition, the sensitivities or relative importance of the random variables is computed.

The random variables statistics are shown in Table 1. The statistics are chosen to be of a typical steel.

Variable	Mean	Standard Deviation
X_1	39.007 ksi	4.0 ksi
X_2	44.326 ksi	6.5 ksi
X_3	10. ksi	2.0 ksi
X_4	50. ksi	5.0 ksi
X_5	30,000. ksi	1,500. ksi
X_6	0.3	0.
X_7	0.5	0.
pressure	19.24	9.61

Table 1

Solution Procedure

The advanced mean value (AMV) solution procedure is used to compute the CDF. This procedure has the advantage of accurately predicting the cumulative distribution function, even for highly nonlinear problems, with a small number of finite element calculations[1,6].

First-order analysis was chosen for this problem. The mean value first order (MVFO) and advanced mean value first-order (AMVFO) methods were used to compute the CDF of the equivalent plastic strain at the inner radius. Because this is a nonlinear analysis, the cylinder is loaded incrementally by 5% load increments.

Step 1 establishes a database of the random variable sensitivities about the mean value. This database was computed for ten increments and therefore can be used when computing the CDF at any of the ten increments. Approximately seven nonlinear finite element solutions were needed for this step, one deterministic, and one for each random variable.

Perturbations of the stress-strain curve are computed by perturbing each random variable independently. The perturbed stress-strain curves are shown in Figure 3.

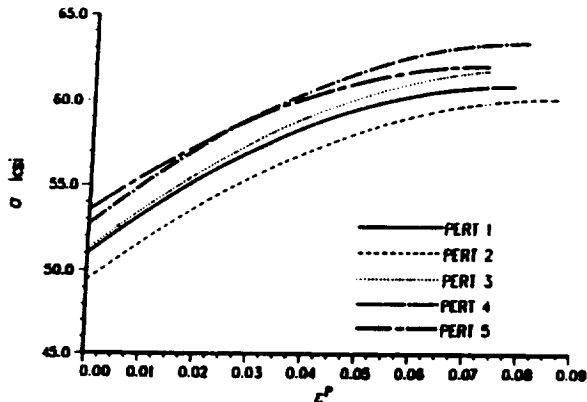


Figure 3. Perturbed Stress-Strain Curves

Step 2 is to compute the MVFO solution and the most probable points (MPP's) at the desired increments by using the sensitivity data and the fast probability method. This procedure does not require finite element analysis and is typically done very quickly. Thus, the MVFO calculation of the CDF can be obtained at a number of increments, structural locations, etc., inexpensively once the sensitivity database has been established.

Step 3 is to update the MVFO solution using the AMVFO method. The AMVFO method involves computing a finite element solution at the MVFO predicted most probable point. Note, the MPP's will in general be different for each increment and each probability level. Thus, if the solution was desired for four increments and ten probability levels, the number of finite element analyses would be, $4 \times 10 = 40$.

Further steps which involve new sensitivity analysis can be taken to check the AMV solution. However, a simple study at one increment showed the AMV method to be accurate.

Results

The CDF results for loading increment 8 are shown in Figure 4. The mean value of equivalent plastic strain at this increment is $5.71E-4$. We see from the figure that the MVFO solution predicts negative equivalent plastic strain for low probabilities. For example, at $u = -5$, or a probability of $2.87E-7$, the predicted equivalent plastic strain is $-.0014$ in/in. Physically, this is not possible. The MVFO solution

is not accurate in this region because it is using sensitivities about the mean value and does not know the physical impossibility of negative equivalent plastic strain.

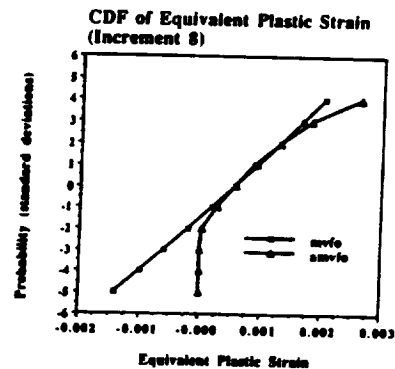


Figure 4. CDF of Equivalent Plastic Strain at Increment 8

The AMVFO method is used to update the MVFO solution and correctly accounts for nonlinearities in the response. Thus, the AMVFO solution predicts zero equivalent plastic strain for low probabilities and correctly predicts the truncated form of the distribution.

Figure 5 shows the cumulative distribution function of the equivalent plastic strain calculated at several load increments, in effect showing the behavior of the CDF as a function of loading. The distributions behave similarly with the truncated effect becoming less pronounced with increasing load as expected.

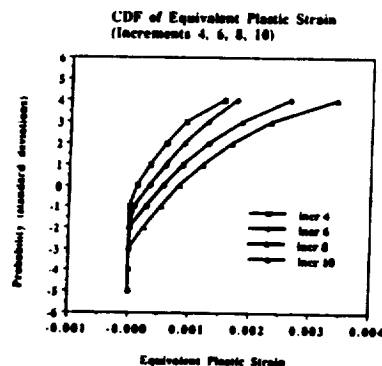


Figure 5. CDF's of Equivalent Plastic Strain at Increments 4, 6, 8, 10

The probabilistic sensitivities are computed by NESSUS at each probability level and each load increment. Figure 6 shows the probabilistic sensitivities at $u = +3$, or probability = .9987, for increments 4 and 10. The random variables, σ_y - yield stress, E - elastic modulus, and initial

load dominate the probabilistic solution. The initial load becomes more dominate as the loading increases. The random variables, x_1 , x_2 , and x_3 are related to the hardening slope, the ultimate stress, and ultimate strain, and have little effect at these low strains.

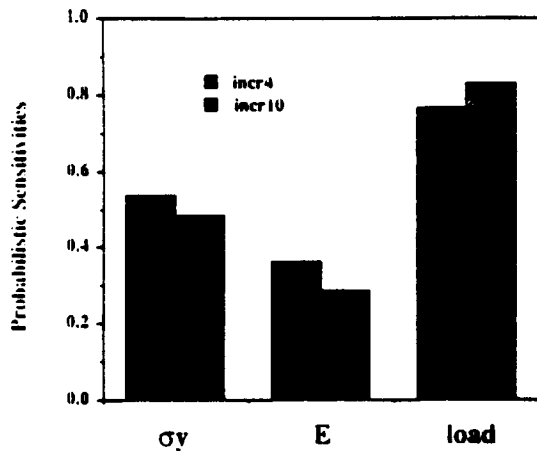


Figure 6. Probabilistic Sensitivity Factors at $u = +3$ ($p = .9987$)

Conclusions

In summary, a method of characterizing random stress-strain behavior with a small number of random variables has been developed. This capability has been integrated into the probabilistic finite element analysis code NESSUS. This system allows the user to perform advanced probabilistic and reliability analysis on complex engineering structures which exhibit nonlinear material behavior.

Acknowledgements

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Appendix

UWKS1 Subroutine

C ... SUBROUTINE UWKS1 ... USER-DEFINED WORKHARDENING SLOPE

C SUBROUTINE UWKS1
1 (RPLAS, SIGT, SLOP, SHIFT, TEMP, MODE)

C *****
C ** DEFINES THE WORKHARDENING SLOPE FOR J2-FLOW PLASTICITY **
C **
C *****

ARGUMENTS:

C RPLAS INPUT THE EQUIVALENT PLASTIC STRAIN VALUE
C SIGT OUTPUT THE YIELD STRESS AT THIS PLASTIC STRAIN
C SLOP OUTPUT THE SLOPE OF THE WORKHARDENING CURVE
C SHIFT OUTPUT THE SHIFT THRESHOLD (BACKSTRESS) VALUE
C TEMP INPUT THE TEMPERATURE AT THE MODE
C MODE INPUT THE MODE NUMBER

NOTES:

- C * THIS SUBROUTINE IS CALLED BY:
 - C YIEL TO CHECK WHETHER PLASTIC YIELDING OCCURS
 - C * THE VALUE OF THE EQUIVALENT PLASTIC STRAIN IS A GIVEN QUANTITY.
C IT WILL BE USED TO INTERPOLATE THE TABLES CODED BY THE USER IN
C THIS SUBROUTINE OR AS AN INPUT FOR A FUNCTIONAL EXPRESSION.
 - C * THE YIELD STRESS AT THIS PLASTIC STRAIN IS USED TO DETERMINE THE
C RADIUS OF THE VON MISES YIELD SURFACE.
 - C * THE SLOPE OF THE WORKHARDENING CURVE IS GIVEN BY
- $$d \sigma_{pl} = \frac{d \sigma_{pl}}{d \epsilon_{pl}}$$
- C WHERE $d \sigma_{pl}$ IS THE STRESS INCREMENT, AND $d \epsilon_{pl}$ IS THE
C CORRESPONDING PLASTIC STRAIN INCREMENT.
- C * THE SHIFT THRESHOLD (BACKSTRESS) VALUE IS THE AMOUNT SUBTRACTED
C FROM THE EQUIVALENT STRESS BEFORE COMPUTING THE CIRCUMFERENCE OF
C THE YIELD SURFACE; IN OTHER WORDS, IT IS THE AMOUNT BY WHICH
C THE YIELD SURFACE IS SHIFTED FROM THE ORIGIN DURING KINEMATIC
C OR CANNING HARDENING; IT IS ZERO FOR ISOTROPIC HARDENING.
 - C * THE TABLES AND/OR FUNCTIONS CODED IN THIS USER SUBROUTINE MAY
C DEPEND ON BOTH SPATIAL LOCATION (MODE) AND THE TEMPERATURE.

C IMPLICIT REAL*8 (A-H,O-S)

C *****
C ** BLOCK CHECK: USER-DEFINED COEFFICIENTS FROM "COEF" OPTION **
C *****

C IUNIT <=> UNIT NUMBER OF MAIN OUTPUT FILE (FOR ANY USER-DEFINED
C OUTPUT THAT MAY BE GENERATED)
C JPERT <=> CURRENT PERTURBATION NUMBER
C (0 .LE. JPRT .LE. NUMBER OF PERTURBATIONS)
C NCOEF <=> NUMBER OF USER-DEFINED COEFFICIENTS (UP TO 10 ALLOWED)
C VCOEF <=> N1's, VALUES OF THE USER-DEFINED COEFFICIENTS FOR
C PERTURBATION 0 JPRT

C COMMON / USRCOF / IUNIT, JPRT, NCOEF, VCOEF(10)

C DIMENSION X(7)

C CUBIC STRESS-STRAIN MODEL

C C0
C =0.0+00
C Q1
C =1.0+00
C Q2
C =2.0+00
C Q3
C =3.0+00
C Q4
C =4.0+00
C Q5
C =5.0+00
C QSMALL=1.E-07

C SET THE PARAMETERS OF THE STRESS-STRAIN CURVE MODEL

C DO 1 I=1,7

1 X(I)=VCOEF(I)

C C compute physical quantities
C
C S = -X(5)
C SLOP=X(4)
C S0 =Q3*(X(1)+X(2))
C SLOP=X(3)+X(4)
C EPSU=X(3)/(X(1)+Q2*X(2))+X(4)/X(5)
C EPS =X(4)
C X0=X(7)

C C FOR LARGE PLASTIC STRAINS BEYOND THE RANGE OF THE STRESS-STRAIN
C CURVE MODEL, LIND HARDENING IS ASSUMED

C SLOP=EPSU-SIGU/E
C IF (SLOP-0.0, MAX) THEN
C SIGT=SIGU
C SHIFT=X0*(SIGU-SIG0)
C SLOP=0
C RETURN
C ENDIF

C C CALCULATE THE CURRENT VALUE OF THE YIELD STRESS (SIGT)

C IT0=10000
C IT=0
C SIG=SIG0

C EPS0 = elastic strain
C EPSU = total strain
C S = interpolation factor = 0 at 0 eps, = 1 at eps
C Q1,Q2,Q3 = cubic shape functions
C SIG2 = yield stress estimate

2 EPS0=SIG0/E
C EPSU=EPS0+RPLAS
C S=(EPSU-EPS0)/(EPSU-SIG0)
C I2=1+S
C I3=2+S
C Q1=Q2*I3-Q3*I2+Q1
C Q2=Q2*I3+Q3*I2
C Q3=(EPSU-SIG0)*(I3-Q2*I2+S)/E
C SIG2=SIG0*Q1+SIG0*Q2+SIG0*Q3

C IF (IT.EQ.0) TOL=QSMALL*SIG2

C F=SIG2-SIGT
C IT=IT+1

C IF (F.LE.0.0) THEN

C C convergence

C SIGT=SIG2

C ELSE

C no convergence

C IF (IT.EQ.1000) THEN

C max iterations

C SIGT=SIG2

C CALL LINE(1,1)

C WRITE(IUNIT,3)

C ELSE

C SIGT=SIG2

C GOTO 2

C ENDIF

C ENDIF

C SHIFT FOR KINEMATIC HARDENING (SHIFT)

C SHIFT=X0*(SIGT-SIG0)

C SLOPE OF THE HARDENING CURVE (SLOP)

C EPS0=SIG0/E

C EPSU=EPS0+RPLAS

C S=(EPSU-SIG0)/(EPSU-SIG0)

C I2=1+S

C I3=2+S

C Q1=Q2*I3-Q3*I2+Q1

C Q2=Q2*I3+Q3*I2

C Q3=(EPSU-SIG0)*(I3-Q2*I2+S)/E

C SIG2=SIG0*Q1+SIG0*Q2+SIG0*Q3

C SLOP=S*SLOP/(S-SLOP)

C RETURN

C 3 FORMAT(2X,'---UWKS1--- TROUBLE WITH CONVERGENCE')

C RETURN

C END

APPENDIX C

NESSUS Component Reliability Validation Problems

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Appendix C.1

Displacement of a Tilted Response

NESSUS 5.0 Response Model Validation

1) Displacement Limits

Purpose:

Demonstrate structural reliability with respect to displacement limits.

Response:

The response is the displacement interference between the tip displacement of the beam, node 31, and a displacement limit d_o , which is also random.

$$g = d_o - d_{beam}$$

Note, because the mean value magnitudes of d_o and d_{beam} are negative, the g function is reformulated as

$$g^* = -g = d_{beam} - d_o$$

When using the reformulated function, the probability of failure or $P[g^* < 0]$ is in the left tail as expected.

Model:

A 10 x 2 cantilever beam composed of 20 plane stress elements is shown in Figure 1. The random variables are shown in the table below.

Variable	Mean	St. Dev.	Distribution
End Load	-100 lbf	20 lbf	Normal
Young's Modulus	10.E6 psi	2.E6 psi	Lognormal
Response Limit	-0.025 in	0.0015 in	Normal

Special Features:

Because of the tilted position of the beam, the desired response is the displacement perpendicular to the beam axis as shown below. Thus, the CVARIABLE option is used to rotate the computed displacements immediately after FEM and before probabilistic analysis. Also, note the decomposition of the end load of the beam into x, y components when applied to the beam and in defining the force random variable.

Results:

The results were computed using the reliability algorithm, ZLEVEL, in NESSUS 5.0 and compared to the results using the PLEVEL algorithm. In addition, Monte Carlo results for the analytical equation are shown.

PFEM INPUT DECK(S)

```

*PFEM
*MVDEFINE
  *DATATYPE      0
  *RESPTYPE      1
  *COMP          2
  *PERT          3
1,2,3
  *RANVAR        3
1,2,3
  *COND          0
  *NODE          32
*END
*AMVDEFINE
  *COMP          2
  *NODE          32
  *ITER          5
           0.0025
*END
*ZFDEFINE
  *COMPUTATIONALMODEL 1 2
1,2
  *EXPLICITMODEL 1 1 0
3
  *ZFUNCT        2 0
           the z function is pre-programmed as NESSUS model
2
  *CVARIABLE      1
    TRANSFORMATION 1
           rotate about the x axis by 30 degrees
      1 30.0
      2 0.0
      3 0.0
    END
*END
*RVDEFINE
  *DEFINE        1
ENDLOAD
-100.0 20.0 NORMAL
           negative mean is ok for normal
FORCE
           notice force random variable definition
33 2 .216506
33 1 -.125
32 2 .433013
32 1 -.250
31 2 .216506
31 1 -.125
  *DEFINE        2
EMOD
10.E6 2.E6 LOGNORMAL
PROP 151
1 33 0.0 1.0 0.0 0.0 0.0
  *DEFINE        3
DELTAO
-0.015 0.0015 NORMAL
*PERT 1
1 0.1
*PERT 2
2 0.1
*PERT 3
3 0.1
*END
*END
*FEM
C TILTED CANTILEVER BEAM DISPLACEMENT MODEL
C response is the tip displacement
C note: rotation is needed to compute tilted displacement

```

```

*DISP
*CONS      0
*ELEMENTS  20
151
*NODES     33
*BOUND     6
*FORCE     6
*PRINT
*MONIT     12
*END
*MONITOR
TOTALDISP  NODE  33 COMPONENT  2
TOTALDISP  NODE  33 COMPONENT  1
TOTALDISP  NODE  32 COMPONENT  2
TOTALDISP  NODE  32 COMPONENT  1
TOTALDISP  NODE  31 COMPONENT  2
TOTALDISP  NODE  31 COMPONENT  1
FORCE      NODE  33 COMPONENT  2
FORCE      NODE  33 COMPONENT  1
FORCE      NODE  32 COMPONENT  2
FORCE      NODE  32 COMPONENT  1
FORCE      NODE  31 COMPONENT  2
FORCE      NODE  31 COMPONENT  1
*ITER      0  2
820      1.E-6
*COORDINATES
1  0.000000E+00  0.000000E+00
2 -.500000E+00  0.866025E+00
3 -.100000E+01  0.173205E+01
4  0.866025E+00  0.500000E+00
5  0.366025E+00  0.136602E+01
6 -.133975E+00  0.223205E+01
7  0.173205E+01  0.100000E+01
8  0.123205E+01  0.186602E+01
9  0.732050E+00  0.273205E+01
10 0.259807E+01  0.150000E+01
11 0.209807E+01  0.236602E+01
12 0.159807E+01  0.323205E+01
13 0.346410E+01  0.200000E+01
14 0.296410E+01  0.286602E+01
15 0.246410E+01  0.373205E+01
16 0.433012E+01  0.250000E+01
17 0.383012E+01  0.336602E+01
18 0.333012E+01  0.423205E+01
19 0.519615E+01  0.300000E+01
20 0.469615E+01  0.386602E+01
21 0.419615E+01  0.473205E+01
22 0.606217E+01  0.350000E+01
23 0.556217E+01  0.436602E+01
24 0.506217E+01  0.523205E+01
25 0.692820E+01  0.400000E+01
26 0.642820E+01  0.486602E+01
27 0.592820E+01  0.573205E+01
28 0.779422E+01  0.450000E+01
29 0.729422E+01  0.536602E+01
30 0.679422E+01  0.623205E+01
31 0.866025E+01  0.500000E+01
32 0.816025E+01  0.586602E+01
33 0.766025E+01  0.673205E+01
*ELEMENTS  151
1  28      31      32      29
2  29      32      33      30
3  25      28      29      26
4  26      29      30      27
5  22      25      26      23
6  23      26      27      24

```

7	19	22	23	20
8	20	23	24	21
9	16	19	20	17
10	17	20	21	18
11	13	16	17	14
12	14	17	18	15
13	10	13	14	11
14	11	14	15	12
15	7	10	11	8
16	8	11	12	9
17	4	7	8	5
18	5	8	9	6
19	1	4	5	2
20	2	5	6	3

C
C fix left end

*BOUND

1	1	0.0
1	2	0.0
2	1	0.0
2	2	0.0
3	1	0.0
3	2	0.0

C

*PROP 151

1	33	1.0	10.E6	0.0	0.0	0.0
---	----	-----	-------	-----	-----	-----

C

C total load is 100 at 45 degree angle

*FORCE

33	2	-21.6506
33	1	12.5
32	2	-43.3013
32	1	25.0
31	2	-21.6506
31	1	12.5

*PRINT

TOTAL NODE

STRESS NODE

*END

*FPI

TILTED BEAM

*RVNUM	3
*DATASETS	4
*GFUNCTION	1
*METHOD	1
*ANALTYPE	1
*PRINT	0

*END

*ZLEVELS 1

0.0

*END

PFEM INPUT DECK (CLOSED-FORM)

```

*PFEM
*MVDEFINE
  *DATATYPE      0
  *RESPTYPE      1
  *COMP          2
  *PERT          3
1,2,3
  *RANVAR        3
1,2,3
  *COND          0
  *NODE          31
*END
*AMVDEFINE
  *COMP          2
  *NODE          31
  *ITER
    5          0.0025
*END
*ZFDEFINE
  *EXPLICITMODEL 3
1 2 3
  *ZFUNCT        5 1
    1.0175
*UZFUNC
*END
*RVDEFINE
  *DEFINE        1
ENDLOAD
-100.0  20.0    NORMAL
  *DEFINE        2
EMOD
10.E6   2.E6    LOGNORMAL
  *DEFINE        3
DELTA0
-0.015   0.0015  NORMAL
*PERT    1
1  0.1
*PERT    2
2  0.1
*PERT    3
3  0.1
*END
*END
*FPI
TILTED BEAM
*RVNUM      3
*DATASETS   4
*GFUNCTION  1
*METHOD    1
*ANALTYPE   1
*PRINT      0
*END
*ZLEVELS    1
0.0
*END

```

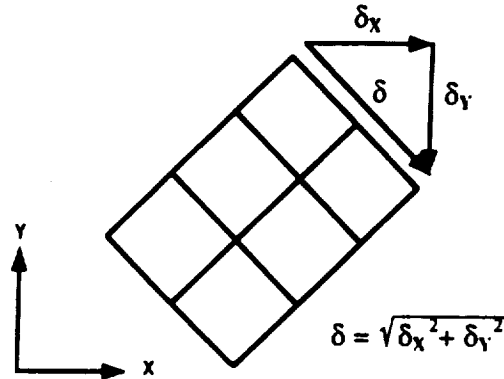


Figure 1.

Deterministic Solution of Tip Displacement

The tip displacement of the beam should be close to the analytical solution for the Euler-Bernoulli beam

$$\delta = \frac{PL^3}{3EI} = \frac{(-100)10^3}{3(10E6)(1/12)(1)(2)^3} = -.005 \text{ in.}$$

The computed result is -0.0050875 in. The FEM model includes shear deflection whereas the Euler-Bernoulli solution does not.

The reliability computed using the AMV based reliability algorithm with a finite element model and with the closed form equations, and the Monte Carlo solution on the closed-form equation is shown below.

METHOD	U	P _r
MVFO	-4.971	3.336 E-7
Quad Est 1	-4.279	9.393 E-6
Quad Est 2	-4.234	1.149 E-5
Iteration 1	-3.969	3.610 E-5
Iteration 2	-3.823	6.594 E-5
Iteration 3	-3.812	6.895 E-5
Iteration 4	-3.811	6.923 E-5
Monte Carlo, #25000, Harbitz Method	-3.783	7.750 E-5
PFEM closed-form	-3.811	6.923 E-5
FPI gfunct 6	-3.708	1.045 E-4

PFEM INPUT DECK(S)

```

*PFEM
*MVDEFINE
  *DATATYPE      0
  *RESPTYPE      1
  *COMP          2
  *PERT          3
1,2,3
  *RANVAR        3
1,2,3
  *COND          0
  *NODE          32
*END
*AMVDEFINE
  *COMP          2
  *NODE          32
  *ITER          5
           0.0025
*END
*ZFDEFINE
  *COMPUTATIONALMODEL 1 2
1,2
  *EXPLICITMODEL 1 1 0
3
  *ZFUNCT        2 0
           the z function is pre-programmed as NESSUS model
2
  *CVARIABLE      1
    TRANSFORMATION 1
      1  30.0
      2  0.0
      3  0.0
    rotate about the x axis by 30 degrees
  END
*END
*RVDEFINE
  *DEFINE        1
ENDLOAD
-100.0  20.0  NORMAL
           negative mean is ok for normal
FORCE
           notice force random variable definition
33  2  .216506
33  1  -.125
32  2  .433013
32  1  -.250
31  2  .216506
31  1  -.125
  *DEFINE        2
EMOD
10.E6  2.E6  LOGNORMAL
PROP  151
1  33  0.0  1.0  0.0  0.0  0.0
  *DEFINE        3
DELTAO
-0.015  0.0015  NORMAL
*PERT  1
1  0.1
*PERT  2
2  0.1
*PERT  3
3  0.1
*END
*END
*FEM
C TILTED CANTILEVER BEAM DISPLACEMENT MODEL
C response is the tip displacement
C note: rotation is needed to compute tilted displacement

```

```

*DISP
*CONS      0
*ELEMENTS  20
151
*NODES     33
*BOUND      6
*FORCE      6
*PRINT
*MONIT      12
*END
*MONITOR
TOTALDISP  NODE  33 COMPONENT  2
TOTALDISP  NODE  33 COMPONENT  1
TOTALDISP  NODE  32 COMPONENT  2
TOTALDISP  NODE  32 COMPONENT  1
TOTALDISP  NODE  31 COMPONENT  2
TOTALDISP  NODE  31 COMPONENT  1
FORCE      NODE  33 COMPONENT  2
FORCE      NODE  33 COMPONENT  1
FORCE      NODE  32 COMPONENT  2
FORCE      NODE  32 COMPONENT  1
FORCE      NODE  31 COMPONENT  2
FORCE      NODE  31 COMPONENT  1
*ITER      0  2
820        1.E-6
*COORDINATES
1  0.000000E+00  0.000000E+00
2  -.500000E+00  0.866025E+00
3  -.100000E+01  0.173205E+01
4  0.866025E+00  0.500000E+00
5  0.366025E+00  0.136602E+01
6  -.133975E+00  0.223205E+01
7  0.173205E+01  0.100000E+01
8  0.123205E+01  0.186602E+01
9  0.732050E+00  0.273205E+01
10 0.259807E+01  0.150000E+01
11 0.209807E+01  0.236602E+01
12 0.159807E+01  0.323205E+01
13 0.346410E+01  0.200000E+01
14 0.296410E+01  0.286602E+01
15 0.246410E+01  0.373205E+01
16 0.433012E+01  0.250000E+01
17 0.383012E+01  0.336602E+01
18 0.333012E+01  0.423205E+01
19 0.519615E+01  0.300000E+01
20 0.469615E+01  0.386602E+01
21 0.419615E+01  0.473205E+01
22 0.606217E+01  0.350000E+01
23 0.556217E+01  0.436602E+01
24 0.506217E+01  0.523205E+01
25 0.692820E+01  0.400000E+01
26 0.642820E+01  0.486602E+01
27 0.592820E+01  0.573205E+01
28 0.779422E+01  0.450000E+01
29 0.729422E+01  0.536602E+01
30 0.679422E+01  0.623205E+01
31 0.866025E+01  0.500000E+01
32 0.816025E+01  0.586602E+01
33 0.766025E+01  0.673205E+01
*ELEMENTS  151
1  28  31  32  29
2  29  32  33  30
3  25  28  29  26
4  26  29  30  27
5  22  25  26  23
6  23  26  27  24

```

7	19	22	23	20
8	20	23	24	21
9	16	19	20	17
10	17	20	21	18
11	13	16	17	14
12	14	17	18	15
13	10	13	14	11
14	11	14	15	12
15	7	10	11	8
16	8	11	12	9
17	4	7	8	5
18	5	8	9	6
19	1	4	5	2
20	2	5	6	3

C
C fix left end
*BOUND
1 1 0.0
1 2 0.0
2 1 0.0
2 2 0.0
3 1 0.0
3 2 0.0
C
*PROP 151
1 33 1.0 10.E6 0.0 0.0 0.0
C
C total load is 100 at 45 degree angle
*FORCE
33 2 -21.6506
33 1 12.5
32 2 -43.3013
32 1 25.0
31 2 -21.6506
31 1 12.5
*PRINT
TOTAL NODE
STRESS NODE
*END
*FPI
TILTED BEAM
*RVNUM 3
*DATASETS 4
*GFUNCTION 1
*METHOD 1
*ANALTYPE 1
*PRINT 0
*END
*ZLEVELS 1
0.0
*END

PFEM INPUT DECK (CLOSED-FORM)

```

*PFEM
*MVDEFINE
  *DATATYPE      0
  *RESPTYPE      1
  *COMP          2
  *PERT          3
1,2,3
  *RANVAR        3
1,2,3
  *COND          0
  *NODE          31
*END
*AMVDEFINE
  *COMP          2
  *NODE          31
  *ITER
    5          0.0025
*END
*ZFDEFINE
  *EXPLICITMODEL 3
1 2 3
  *ZFUNCT        5  1
    1.0175
*UZFUNC
*END
*RVDEFINE
  *DEFINE        1
ENDLOAD
-100.0  20.0    NORMAL
  *DEFINE        2
EMOD
10.E6  2.E6    LOGNORMAL
  *DEFINE        3
DELTAO
-0.015  0.0015  NORMAL
*PERT      1
1  0.1
*PERT      2
2  0.1
*PERT      3
3  0.1
*END
*END
*FPI
TILTED BEAM
*RVNUM      3
*DATASETS   4
*GFUNCTION  1
*METHOD    1
*ANALTYPE   1
*PRINT      0
*END
*ZLEVELS    1
0.0
*END

```

Appendix C.2

Probabilistic Buckling Analysis of a Simply-Supported Beam

Buckling Limits

Introduction

This example demonstrates the computation of structural reliability with respect to buckling limits. The probability of the first buckling load of a simply supported beam exceeding a user-defined limit is computed and compared with the exact solution.

Analytical or Exact Solution

The limit state, or g function, is defined as

$\lambda_b - \lambda_o$, where λ_o is a user-defined buckling limit and λ_b is the buckling load of the beam.

The Euler-Bernoulli solution for the buckling load is $\lambda_b = \frac{\pi^2 EI}{L^2}$

Reference:

G. J. Simitses, An Introduction to the Elastic Stability of Structures, Prentice-Hall, 1976.

Modeling and Methods

A 10-in. simply supported beam composed of 20 beam elements, type 98, is shown below. The cross-section is circular with inner and outer radii. The random variables are shown below.

Variable	Mean	St. Dev.	Distribution
Inner Radius	.1	.015	Normal
Outer Radius	.2	.02	Normal
Beam Length	10	.5	Lognormal
Buckling Limit	3800	0.1	Normal

Notice that the variance of the buckling limit is sufficiently small as to be considered deterministic.

The AMV iteration tolerance was set at 0.25%. The FEM iteration tolerance was also set at 0.25%.

Results and Discussion

The results were computed using the reliability algorithm, ZLEVEL. The simple response function $z = \lambda - \lambda_o$ is programmed into NESSUS as model 1 and was used for this analysis. The reliability solutions using the closed-form equations were obtained using three different methods: Monte Carlo, FPI (Gfunction 6), and PFEM. Convergence was obtained using ZLEVEL after 2 iterations. The closed-form equations were adjusted by the ratio of the NESSUS/Theory buckling loads computed at the mean value.



Deterministic Response

METHOD	U	P _r
MVFO	-4.971	3.336 E-7
Quad Est 1	-3.662	1.252 E-4
Quad Est 2	-3.079	1.039 E-3
Iteration 1	-3.137	8.535 E-4
Iteration 2	-3.135	8.593 E-4
Monte Carlo, #25000, Harbitz Method	-3.155	8.026 E-4
PFEM closed-form	-3.130	8.741 E-4
FPI gfunct 6	-3.103	9.579 E-4

PFEM Input Decks

PFEM input decks for ZLEVELS solutions using finite element model and closed form equations are shown below.

PFEM input deck - FEM model

```

*PFEM
*MVDEFINE
  *COND          1          1st eigenvalue
  *DATATYPE      1          modal analysis (buckling)
  *RESPTYPE      30         response is the eigenvalue
  *COMP          1 1        has no meaning for eigenvalue
  *NODE          1 1        has no meaning for eigenvalue
  *PERT          4
1,2,3,4
  *RANVAR          4
1,2,3,4
C  *REST          1
*END
*AMVDEFINE
  *COND          1 1
  *COMP          1 1
  *NODE          1 1
  *ITER          5          0.0025
*END
*ZFDEFINE
  *COMPUTATIONALMODEL 1 3
1,2,3
  *EXPLICITMODEL 1
4
  *ZFUNCT          1 0          uses {do - d} as zfunction
*END
C-----
*RVDEFINE
  *DEFINE 1
RO
  0.2 0.02          NORMAL
  BEAMSECTION      3
1 21 1.0 0.0
C-----
  *DEFINE 2
RI
  0.1 0.015          NORMAL
  BEAMSECTION      3
1 21 0.0 1.0
C-----
  *DEFINE          3
LENGTH
  10.00 0.5          LOGNORMAL
  COORDINATES
1 0.0 0.0 0.0 0.0 0.0 0.0
2 0.0 .05 0.0 0.0 0.0 0.0
3 0.0 .10 0.0 0.0 0.0 0.0
4 0.0 .15 0.0 0.0 0.0 0.0
5 0.0 .20 0.0 0.0 0.0 0.0
6 0.0 .25 0.0 0.0 0.0 0.0
7 0.0 .30 0.0 0.0 0.0 0.0
8 0.0 .35 0.0 0.0 0.0 0.0
9 0.0 .40 0.0 0.0 0.0 0.0
10 0.0 .45 0.0 0.0 0.0 0.0
11 0.0 .50 0.0 0.0 0.0 0.0
12 0.0 .55 0.0 0.0 0.0 0.0
13 0.0 .60 0.0 0.0 0.0 0.0
14 0.0 .65 0.0 0.0 0.0 0.0

```

15	0.0	.70	0.0	0.0	0.0	0.0
16	0.0	.75	0.0	0.0	0.0	0.0
17	0.0	.80	0.0	0.0	0.0	0.0
18	0.0	.85	0.0	0.0	0.0	0.0
19	0.0	.90	0.0	0.0	0.0	0.0
20	0.0	.95	0.0	0.0	0.0	0.0
21	0.0	1.0	0.0	0.0	0.0	0.0

```

*DEFINE 4
BLIMIT
3800. 0.1 NORMAL
*PERT 1
1 0.1
*PERT 2
2 0.1
*PERT 3
3 0.1
*PERT 4
4 0.1
*END
*END
C-----
*FEM
C - SOA BUCKLING MODEL - PINNED COLUMN
*DISP
*CONS 0
*BUCKLE 1 2 1
*FORCE 2
*BOUND 12
*BEAMSECTION 1
*ELEMENTS 20
98
*NODES 21
*PRINT
*END
*ITER 0 3
200 0.005
*COORDINATES
1 0.0 0.0 0.0 1.0 0.0 0.0
2 0.0 0.5 0.0 1.0 0.0 0.0
3 0.0 1.0 0.0 1.0 0.0 0.0
4 0.0 1.5 0.0 1.0 0.0 0.0
5 0.0 2.0 0.0 1.0 0.0 0.0
6 0.0 2.5 0.0 1.0 0.0 0.0
7 0.0 3.0 0.0 1.0 0.0 0.0
8 0.0 3.5 0.0 1.0 0.0 0.0
9 0.0 4.0 0.0 1.0 0.0 0.0
10 0.0 4.5 0.0 1.0 0.0 0.0
11 0.0 5.0 0.0 1.0 0.0 0.0
12 0.0 5.5 0.0 1.0 0.0 0.0
13 0.0 6.0 0.0 1.0 0.0 0.0
14 0.0 6.5 0.0 1.0 0.0 0.0
15 0.0 7.0 0.0 1.0 0.0 0.0
16 0.0 7.5 0.0 1.0 0.0 0.0
17 0.0 8.0 0.0 1.0 0.0 0.0
18 0.0 8.5 0.0 1.0 0.0 0.0
19 0.0 9.0 0.0 1.0 0.0 0.0
20 0.0 9.5 0.0 1.0 0.0 0.0
21 0.0 10.0 0.0 1.0 0.0 0.0
*ELEMENTS 98
1 1 2
2 2 3
3 3 4
4 4 5
5 5 6
6 6 7
7 7 8

```

8	8	9
9	9	10
10	10	11
11	11	12
12	12	13
13	13	14
14	14	15
15	15	16
16	16	17
17	17	18
18	18	19
19	19	20
20	20	21

```

*PROP      98
1  21  1.0  10.E6  0.0
*BEAMSECTION  3
1  21  0.2  0.1
*BOUND
1  1  0.0
1  2  0.0
1  3  0.0
1  4  0.0
1  5  0.0
21 1  0.0
21 3  0.0
21 4  0.0
21 5  0.0
*FORCE
21 2  -1.0
*PRINT
TOTAL NODE
STRESS NODE
*END
C
C ... INCREMENT 1 ( BUCKLING ANALYSIS )
*END
*STOP
*FPI
TILTED BEAM
*RVNUM      4
*DATASETS    5
*GFUNCTION   1
*METHOD     1
*ANALTYPE    1
*PRINT       0
*END
*ZLEVELS     1
0.0
*END

```

PFEM Input Deck - Closed-Form Model

```

*PFEM
*MVDEFINE
  *COND          1  1          1st eigenvalue
  *DATATYPE      1            modal analysis (buckling)
  *RESPTYPE      30          response is the eigenvalue
  *COMP          1  1        has no meaning for eigenvalue
  *NODE          1  1        has no meaning for eigenvalue
  *PERT          4
1,2,3,4
  *RANVAR        4
1,2,3,4
C  *REST          1
*END
*AMVDEFINE
  *COND          1  1
  *COMP          1  1
  *NODE          1  1
  *ITER          5          0.0025
*END
*ZFDEFINE
  *EXPLICITMODEL 4
1,2,3,4
  *ZFUNCT        6  2
1
1.001675
  *UZFUNC
*END
C-----
*RVDEFINE
  *DEFINE 1
RO
  0.2  0.02          NORMAL
C-----
  *DEFINE 2
RI
  0.1  0.015         NORMAL
C-----
  *DEFINE 3
LENGTH
  10.00  0.5         LOGNORMAL
  *DEFINE 4
BLIMIT
3800.  0.1          NORMAL
  *PERT          1
1  0.1
  *PERT          2
2  0.1
  *PERT          3
3  0.1          notice small perturbation for imperfection
  *PERT          4
4  0.1
*END
*END
C-----
*FPI
TILTED BEAM
*RVNUM          4
*DATASETS       5
*GFUNCTION       1
*METHOD         1
*ANALTYPE        1
*PRINT           0
*END

```

*ZLEVELS 1
0.0
*END

1 Stress Response of a Circular Disk

1.1 Introduction

A circular disk is analyzed with two equal and opposite forces P acting along a diameter. Failure is assumed when the maximum compressive stress σ due to the point loads exceeds some limiting stress σ_0 . Therefore, the probability of failure is given by,

$$P_f = P[\sigma_0 \leq \sigma].$$

Once P_f is computed, the reliability is given as $1-P_f$.

1.2 Analytical Solution

For point loading acting along the y-axis, the maximum compressive stress is located at the center of the disk and is given by[1],

$$\sigma_r = -\frac{6P}{\pi d t}$$

where P is the point load, d is the diameter of the disk, and t is the thickness.

1.3 Modeling and Methods

A one-quarter symmetry model is used for the finite element calculations. A representation of the disk is shown in Figure 1-1 and the finite element mesh is shown in Figure 1-2. Plane stress conditions are assumed, so NESSUS type 151 elements are used. Symmetry boundary conditions are imposed along the $x=0$ and $y=0$ coordinate lines. The error between the exact and finite element solution is,

$$\frac{-3.01863}{-3.0} = 1.02721 \quad (+2.7\%)$$

To enable comparison, this factor is used during the probabilistic calculations to adjust the closed-form solution to match the finite element solution.

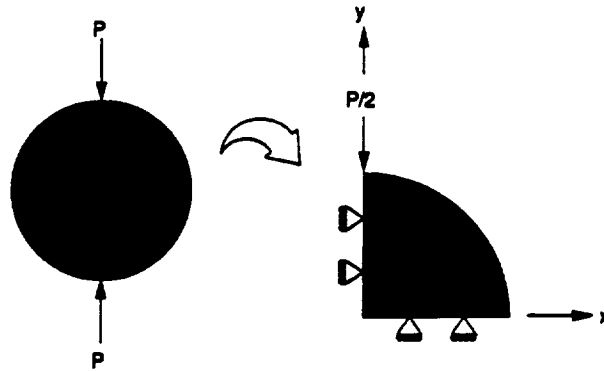


Figure 1-1 Circular Disk Problem

Random variables include the point load acting at the center of the disk P , the thickness t , and the limiting stress σ_0 . Table 1-1 lists the input parameters used for these random variables.

Table 1-1. Random Variables Used in the Circular Disk Demonstration Problem

Variable	Description	Mean	Standard Deviation	Distribution
P	Point Load	10π lb.	π lb.	Normal
t	Thickness	1.0 in.	0.1 in.	Normal
σ_0	Limiting Stress	-5.0 psi	0.5 psi	Normal

The NESSUS input deck for the PFEM analysis is shown in Listing 1-1. The closed-form solution for the stress is coded in the QRPLAT subroutine, shown in Listing 1-2, as is the Z-function $Z = \sigma_0 - \sigma$. This model is selected as model 2 on the *ZFUNCTION keyword card. Note that to compute the stress using the closed-form expression rather than by finite elements only requires the *ZFDEFINE input block, shown in Listing 1-1, to be as defined as follows:

```
*ZFDEFINE
*EXPLICITVARIABLES 3
  1 2 3
*ZFUNCTION 2 1
  1.02721
*UZFUNCTION
*END
```

Here, the coefficient 1.02721 is the factor between the exact and computed stress.

In the *ZFDEFINE block shown in Listing 1-1, a *CVARIABLE (computational variable) is used. In this input block, CVARIABLE number 1 is defined as the resultant of OPERATION 2 on the response values for NODES 1, 2, 3, 4, 5, 6, and 7. OPERATION 2 is a pre-defined NESSUS data operation to return the algebraic minimum value, which in this case, corresponds to the maximum *negative* stress. The response quantity for the seven nodes is chosen to be the stress, component 2, at condition (increment) 0, and is selected in the *MVDEFINE block.

The input deck for the Monte Carlo simulation is given in Listing 1-3.

1.4 Results and Discussion

To verify the point probability of failure computation, an entire "cdf" of probability of failure was computed using Monte Carlo simulation with 100000 samples (Figure 1-3). Figure 1-4 gives a close-up view of the point probability of failure results. Table 1-2 gives the results in tabular form.

Table 1-2 Results from the Circular Disk Demonstration Problem

Method	Probability of Failure
Monte Carlo (N=100000)	0.0041226
FPI (Closed-Form, GFUNC=6)	0.0044202
AMV+ (Closed-Form, GFUNC=1)	0.0037154
AMV+ (NESSUS/FEM, GFUNC=1)	0.0037215

1.5 References

1. Timoshenko, S.P. and Goodier, J. N., *Theory of Elasticity*, McGraw Hill, 3rd ed., pg. 122, 1951.

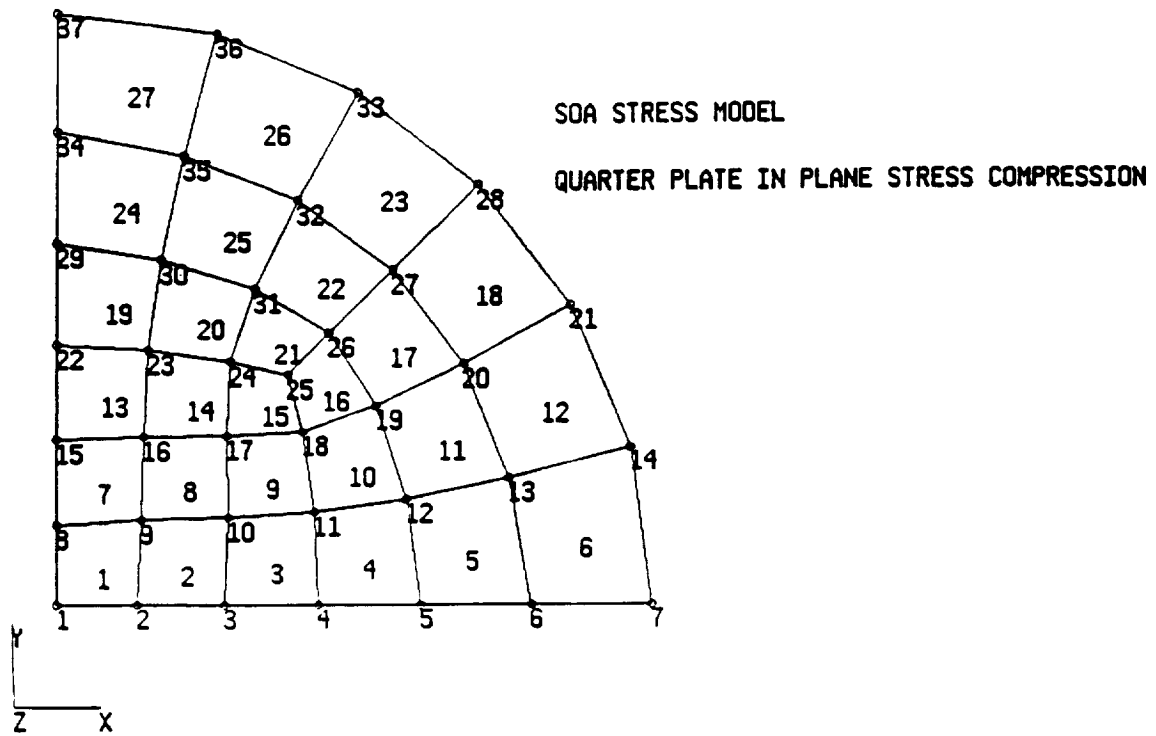
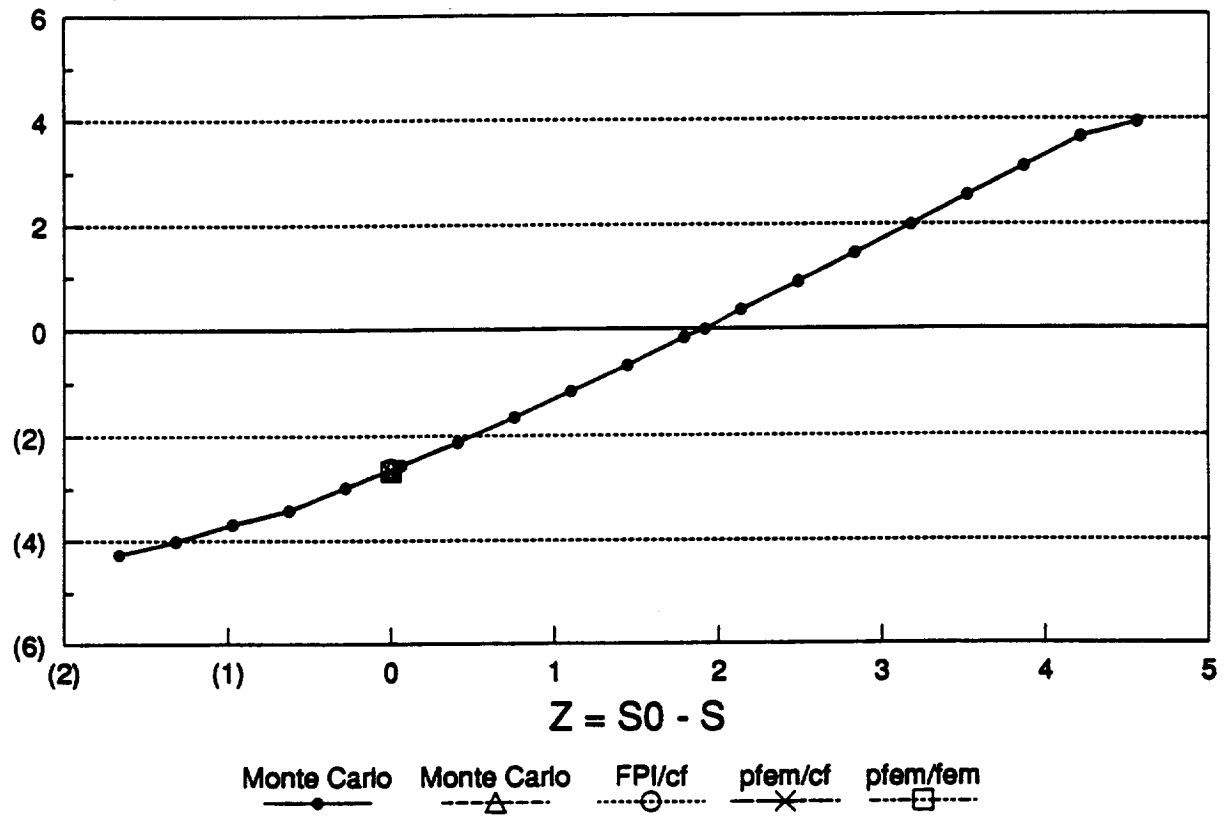


Figure 1-2 Finite Element Mesh of the Circular Disk Problem

CDF (Standard Deviations)



CDF (Standard Deviations)

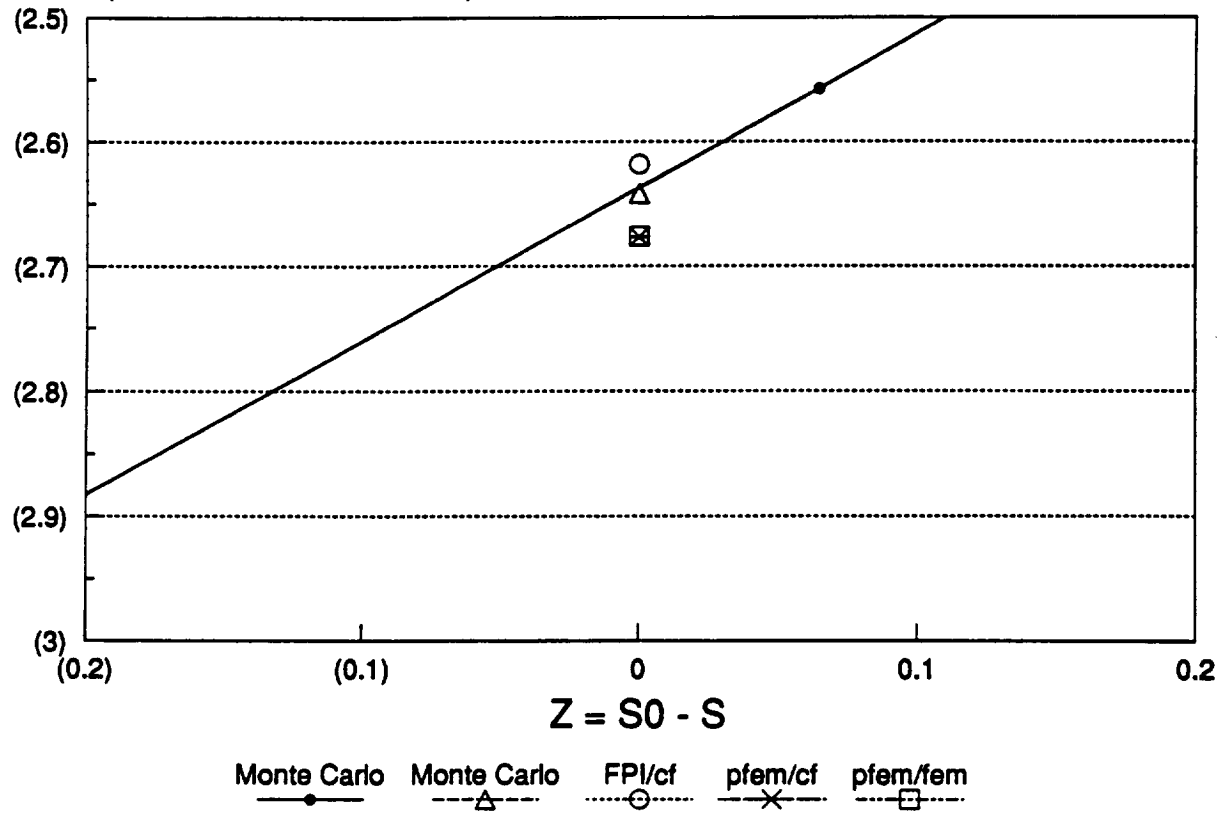


Figure 1-4 Probability of Failure Results for the Circular Disk Problem

Listing 1-1 NESSUS/PFEM Input Deck for the Circular Disk Problem

```

*PFEM
C
C Z-FUNCTION IS DEFINED BY FEM CODE AND 1 EXPLICIT VARIABLE.
C DESIGN FACTOR MODEL 2 IS USED.
C CVariable IS THE MINIMUM FROM NODES 1-7 (DATA DEFINED IN *MVDEFINE)
C
*ZFDEFINE
  *COMPUTATIONALMETHOD 1 2
  1 2
  *EXPLICITVARIABLES 1
  3
  *ZFUNCT 2 0
  *CVariable 1
  NODELIST 7
  1 2 3 4 5 6 7
  OPERATION 2
  END
  *END
C
C THREE RANDOM VARIABLES
C
*RVDEFINE
  *DEFINE 1
  LOAD
  -15.70796327 1.570796327 NORMAL
  FORCE
  37 2 1.0
  *DEFINE 2
  THICK
  1.0 0.1 NORMAL
  PROP 151
  1 37 1.0 0.0 0.0 0.0 0.0
  *DEFINE 3
  MAXSTRES
  -5.0 0.5 NORMAL
  *PERT 1
  1 0.1
  *PERT 2
  2 0.1
  *PERT 3
  3 0.1
  *END
C
C MV ANALYSIS. INCREMENTAL DATA, STRESS RESPONSE, COMPONENT 2 (Y). NODELIST
C IS USED (SEE CVariable IN ZFDEFINE) TO SEARCH FOR MAX STRESS.
C
*MVDEFINE
  *DATATYPE 0
  *RESPTYPE 3
  *COND 0

```

```

*COMP 2
*PERT 3
  1 2 3
*RANVAR 3
  1 2 3
*END
C
C AMV ANALYSIS
C
*AMVDEFINE
  *NODE 1
  *COMPONENT 2
  *ITERATION
    10 0.005
  *END
*END
*FEM
C SOA STRESS MODEL - QUARTER ROUND PLATE IN PLANE STRESS COMPRESSION
C
C PARAMETER INPUT
C
*DISPLACEMENTMETHOD
*CONSTITUTIVE 0
*ELEMENTS 27
  151
*NODES 37
*BOUNDARY 14
*FORCE 1
*PRINT
*MONITOR 14
*END
C
C MODEL DATA INPUT
C
*ITER 0 2
  20 0.01
*COORDINATES
  1 0.000000E+00 0.000000E+00
  2 0.134787E+01 0.000000E+00
  3 0.280959E+01 0.000000E+00
  4 0.439479E+01 0.000000E+00
  5 0.611389E+01 0.000000E+00
  6 0.797821E+01 0.000000E+00
  7 0.100000E+02 0.000000E+00
  8 0.000000E+00 0.134787E+01
  9 0.141905E+01 0.142403E+01
 10 0.286135E+01 0.147480E+01
 11 0.432867E+01 0.156613E+01
 12 0.587043E+01 0.178268E+01
 13 0.759568E+01 0.213750E+01
 14 0.964279E+01 0.265840E+01
 15 0.000000E+00 0.280959E+01
 16 0.145117E+01 0.285388E+01
 17 0.284501E+01 0.285999E+01

```

18	0.412128E+01	0.294152E+01
19	0.536625E+01	0.336459E+01
20	0.682626E+01	0.406627E+01
21	0.863470E+01	0.504719E+01
22	0.000000E+00	0.439479E+01
23	0.152485E+01	0.430128E+01
24	0.289885E+01	0.411028E+01
25	0.387187E+01	0.389056E+01
26	0.456389E+01	0.457995E+01
27	0.563832E+01	0.564813E+01
28	0.707107E+01	0.707106E+01
29	0.000000E+00	0.611389E+01
30	0.173691E+01	0.583001E+01
31	0.331194E+01	0.533609E+01
32	0.403561E+01	0.680794E+01
33	0.504719E+01	0.863470E+01
34	0.000000E+00	0.797821E+01
35	0.211085E+01	0.756878E+01
36	0.265840E+01	0.964279E+01
37	0.000000E+00	0.100000E+02

*ELEMENTS 151

1	1	2	9	8
2	2	3	10	9
3	3	4	11	10
4	4	5	12	11
5	5	6	13	12
6	6	7	14	13
7	8	9	16	15
8	9	10	17	16
9	10	11	18	17
10	11	12	19	18
11	12	13	20	19
12	13	14	21	20
13	15	16	23	22
14	16	17	24	23
15	17	18	25	24
16	18	19	26	25
17	19	20	27	26
18	20	21	28	27
19	23	30	29	22
20	24	31	30	23
21	25	26	31	24
22	26	27	32	31
23	27	28	33	32
24	30	35	34	29
25	31	32	35	30
26	32	33	36	35
27	35	36	37	34

*PROP 151

1	37	1.00	30.0E+6	0.3
---	----	------	---------	-----

*BOUNDARY

1	1	0.0
1	2	0.0
2	2	0.0

```

      3      2      0.0
      4      2      0.0
      5      2      0.0
      6      2      0.0
      7      2      0.0
      8      1      0.0
     15      1      0.0
     22      1      0.0
     29      1      0.0
     34      1      0.0
     37      1      0.0
*FORCE
      37      2     -15.70796327
*PRINT
TOTAL NODE
STRESS NODE
*MONITOR
STRESS NODE 1  COMPONENT 1
STRESS NODE 2  COMPONENT 1
STRESS NODE 3  COMPONENT 1
STRESS NODE 4  COMPONENT 1
STRESS NODE 5  COMPONENT 1
STRESS NODE 6  COMPONENT 1
STRESS NODE 7  COMPONENT 1
STRESS NODE 1 COMPONENT 2
STRESS NODE 8  COMPONENT 2
STRESS NODE 15 COMPONENT 2
STRESS NODE 22 COMPONENT 2
STRESS NODE 29 COMPONENT 2
STRESS NODE 34 COMPONENT 2
STRESS NODE 37 COMPONENT 2
*END
C
C
C
*FPI
SOA STRESS MODEL - QUARTER ROUND PLATE IN PLANE STRESS COMPRESSION
*RVNUM 3
*DATASETS 4
*GFUNCTION 1
*METHOD 1
*ANALTYPE 1
*PRINT 0
*END
*ZLEVELS 1
      0.0
*END

```

Listing 1-2 QRPLAT Subroutine

```

SUBROUTINE QRPLAT (NFMVR, FEMRES, PLOAD, THICK, SYLIM, CFACT, RADIUS,
+                 VALDV)
IMPLICIT DOUBLE PRECISION (A-H, O-Z)

C
C STRESS RESPONSE OF A ROUND PLATE SUBJECTED TO 2 OPPOSING POINT LOADS
C
C NFMVR - NUMBER OF FEM RESPONSE VARIABLES
C FEMRES - FEM RESPONSE VALUE
C PLOAD - APPLIED LOAD
C THICK - THICKNESS
C SYLIM - LIMITING Y STRESS
C CFACT - FEM/EXACT CORRECTION FACTOR
C RADIUS - RADIUS OF PLATE
C VALDV - RETURNED Z-VALUE
C
C IF NFMVR>0, THE STRESS WAS COMPUTED BY NESSUS/FEM SO USE FEMRES.
C OTHERWISE, COMPUTE STRESS USING CLOSED FORM EQUATION. NOTE THAT IF
C THE STRESS CAME FROM NESSUS/FEM, THE APPLIED LOAD WAS HALVED SINCE
C A SYMMETRY MODEL WAS USED; THEREFORE WE MULTIPLY IT BY TWO HERE.
C
C .....
C
C          3(P)
C  Syy = -----
C          pi(t)(R)
C
C      IF (NFMVR.GT.0) THEN
C          SIGYY = FEMRES
C      ELSE
C          PI = ACOS(-1.0D0)
C          TWO = 2.0D0
C          PLOAD = PLOAD*TWO
C          SIGYY = 3.0D0*PLOAD/(PI*THICK*RADIUS)
C      ENDIF
C
C LIKE DESIGN FACTOR MODEL #2: Z = S - R (RESISTANCE IS ALGEBRAICALLY
C LESS THAN STRESS, SO SWITCH FROM NORMAL CONVENTION)
C
C      VALDV = SIGYY*CFACT - SYLIM
C
C      RETURN
C      END
C

```

Listing 1-3 Input Deck for the Monte Carlo Simulation

```

*FPI
SOA STRESS MODEL - QUARTER ROUND PLATE IN PLANE STRESS COMPRESSION (9/21/90)
*RVNUM 3
*GFUNCTION 6
*METHOD 5
*ANALTYPE 1
*PRINT 0
*END
*ZLEVELS 1
  0.0
*MONTE
  100000, 1234., 0.9
*EXACTPRM
  21, 2, 0
  10.0
  1.02721
*DEFRANV
LOAD
-0.157080E+02      0.157080E+01      0.200000E+01
THICK
  0.100000E+01      0.100000E+00      0.200000E+01
MAXSTRES
-0.500000E+01      0.500000E+00      0.200000E+01
*END

```

1 Fatigue Life of a Three-Point Bend Specimen

1.1 Introduction

The reliability of a three-point bend specimen against failure by fracture is computed. Failure is defined as

$$P_f = P[N_f \leq N_0]$$

where N_f is the computed number of cycles to failure and N_0 is the design life. N_f is based on the number of cycles to advance a crack to a critical length.

1.2 Analytical Solution

The Paris relation,

$$\frac{\partial a}{\partial N} = C(\Delta K)^n$$

is used to describe the crack growth. The crack is grown from an assumed intrinsic flaw size a_i to a critical length a_c at which point failure is assumed to occur, $a_f = a_c$.

Assuming that the stress intensity geometry factor is constant and that the stress range is $\{0 < \sigma < \sigma_{\max}\}$, the Paris relation can be integrated from a_i to a_f to yield,

$$N_f = \frac{2[a_f^{(1-n/2)} - a_i^{(1-n/2)}]}{C(2-n)(Y\sigma_{\max}\sqrt{\pi})^n}, \quad \text{for } n \neq 2$$

where for the three-point bend specimen,

$$\sigma_{\max} = \frac{3PS}{2B^2}.$$

P , S , and B are defined in Figure 1-1.

1.3 Modeling and Methods

The material is 2024-T3 aluminum. The dimensions of the beam have been chosen such that the stress intensity geometry factor is essentially constant.

A representation of the three-point specimen is shown in Figure 1-1. Using symmetry, only one-half of the beam is modeled. Pinned boundary conditions are imposed at either end of the beam and loading consists of a single point load acting downward at the center of the beam. A 2x2 mesh, shown in Figure 1-2, of plane stress (type 151) elements is used to model the beam.

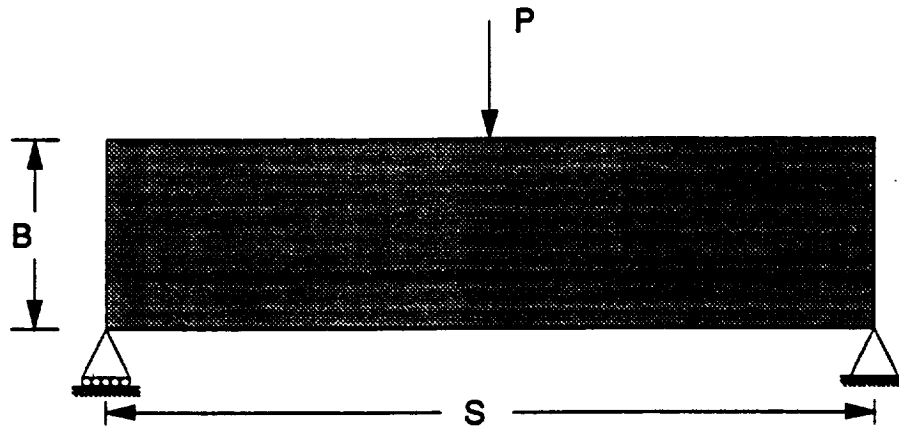


Figure 1-1 Three-Point Bend Specimen

The problem is solved three ways: (i) using the NESSUS AMV+ algorithm with σ_{\max} obtained from a simple finite element model of the three-point bend specimen, (ii) using the NESSUS AMV+ algorithm with σ_{\max} obtained using the closed-form solution, and (iii) using Monte Carlo with σ_{\max} obtained using the closed-form solution.

Random variables include the point load acting at the center of the beam P , the length of the beam S , the height of the beam B , the initial crack length a_i , the final crack length a_f , the Paris coefficient C , the stress intensity geometry factor Y , and the design life N_0 . The Paris exponent n is assumed deterministic, and Y is assumed constant for all crack lengths between a_i and a_f . Table 1-1 lists the input parameters used for these random variables.

**Table 1-1. Random Variables Used in the Three-Point Bend Fatigue
Life Demonstration Problem**

Variable	Description	Mean	Standard Deviation	Distribution
P	Point Load	6.6667 kips	0.6667 kips	Lognormal
S	Beam Length	8.0 in.	0.8 in.	Normal
B	Beam Width	2.0 in.	0.2 in.	Normal
a_f	Final Crack Length	0.09 in.	0.009 in.	Normal
a_i	Initial Crack Length	0.01 in.	0.002 in.	Normal
C	Paris Exponent	1.16 E-09	1.16 E-10	Normal
Y	Stress Intensity Geometry Factor	1.0	0.1	Normal
n	Paris Exponent	4	0	-
N_0	Design Life	20,000 cycles	20 cycles	Normal

The NESSUS input deck for the PFEM analysis is shown in Listing 1-1. The calculation of N_f is coded in the PARISNF subroutine, shown in Listing 1-2, and selected as model 3 on the *ZFUNCTION keyword card. The Paris exponent is passed into PARISNF as a user coefficient, also on the *ZFUNCTION card. Note that to compute the stress using the closed-form expression rather than by finite elements only requires the *ZFDEFINE input block to be as defined as follows:

```
*ZFDEFINE
*EXPLICITVARIABLES 8
 1 2 3 4 5 6 7 8
*ZFUNCTION 3 1
 4.0
*UFUNCTION
*END
```

The input deck for the Monte Carlo simulation is given in Listing 1-3.

1.4 Results and Discussion

To verify the point probability of failure computation, an entire "cdf" of probability of failure was computed using both the NESSUS/PFEM AMV+ procedure and Monte Carlo simulation with 100,000 samples (Figure 1-2). Figures 1-2 and 1-3 give close-up views of the point probability of failure results. Table 1-2 gives the results in tabular form.

Table 1-2 Results from the Three Point Bend Specimen Demonstration Problem

Method	Probability of Failure
AMV+ (NESSUS/FEM)	0.205119
AMV+ (Closed Form)	0.209152
Monte Carlo (N=100000)	0.196020

1.5 References

1. Broke, D., "*Elementary Engineering Fracture Mechanics*, Martinus Nijhoff Publishers, The Hague, 1984.

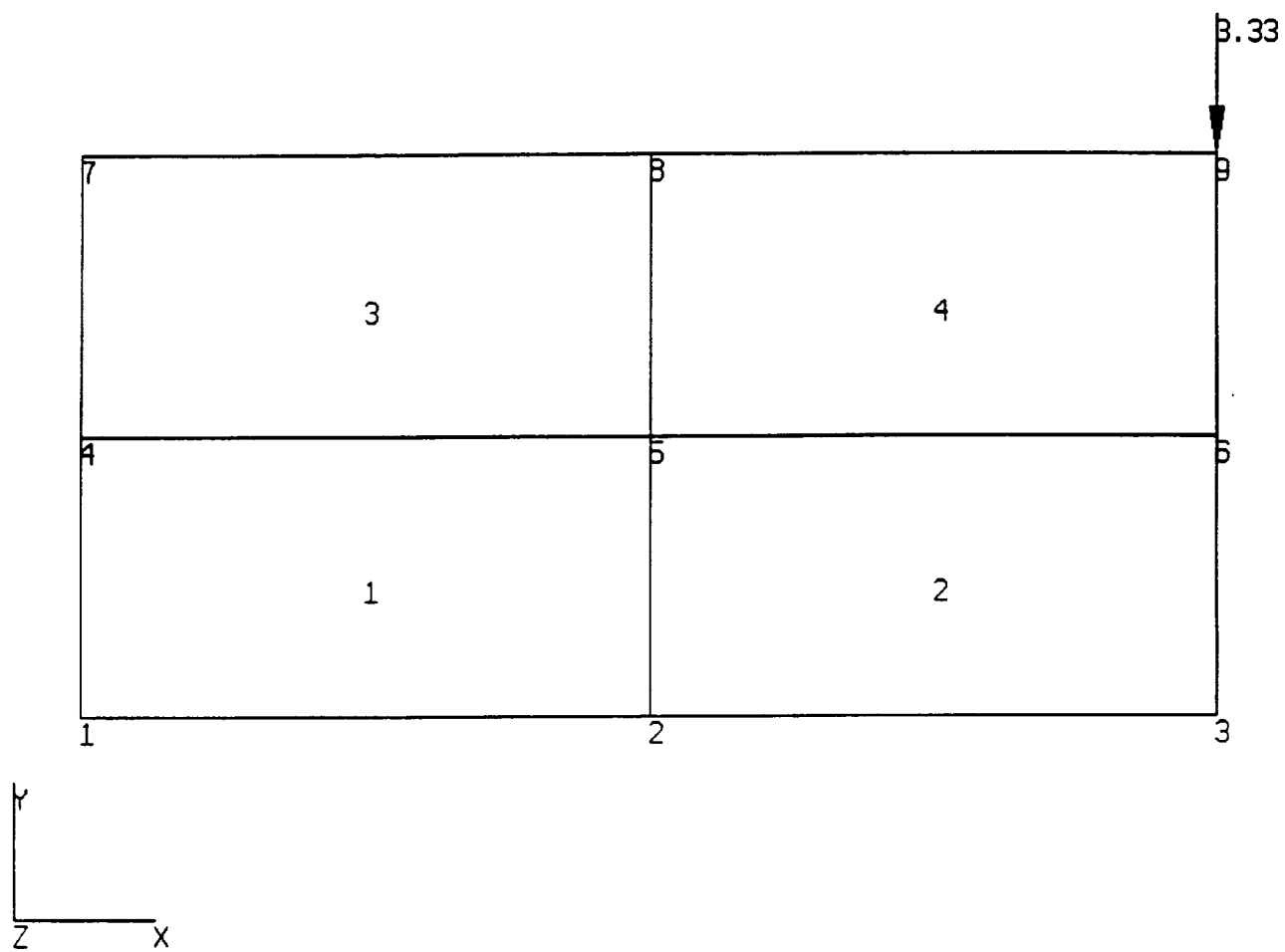


Figure 1-2 Finite Element Mesh of the Three-Point Bend Specimen

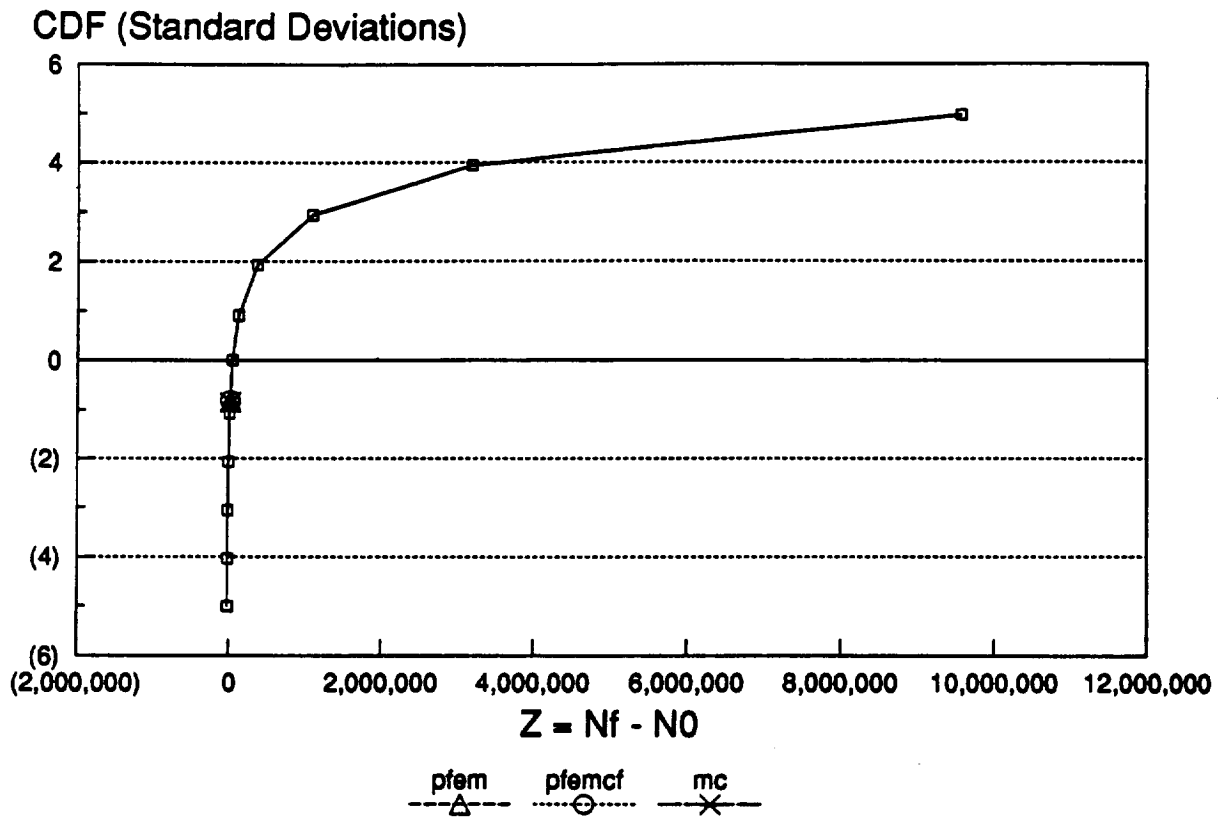


Figure 1-3 Probability of Failure Results for Three-Point Bend Specimen

CDF (Standard Deviations)

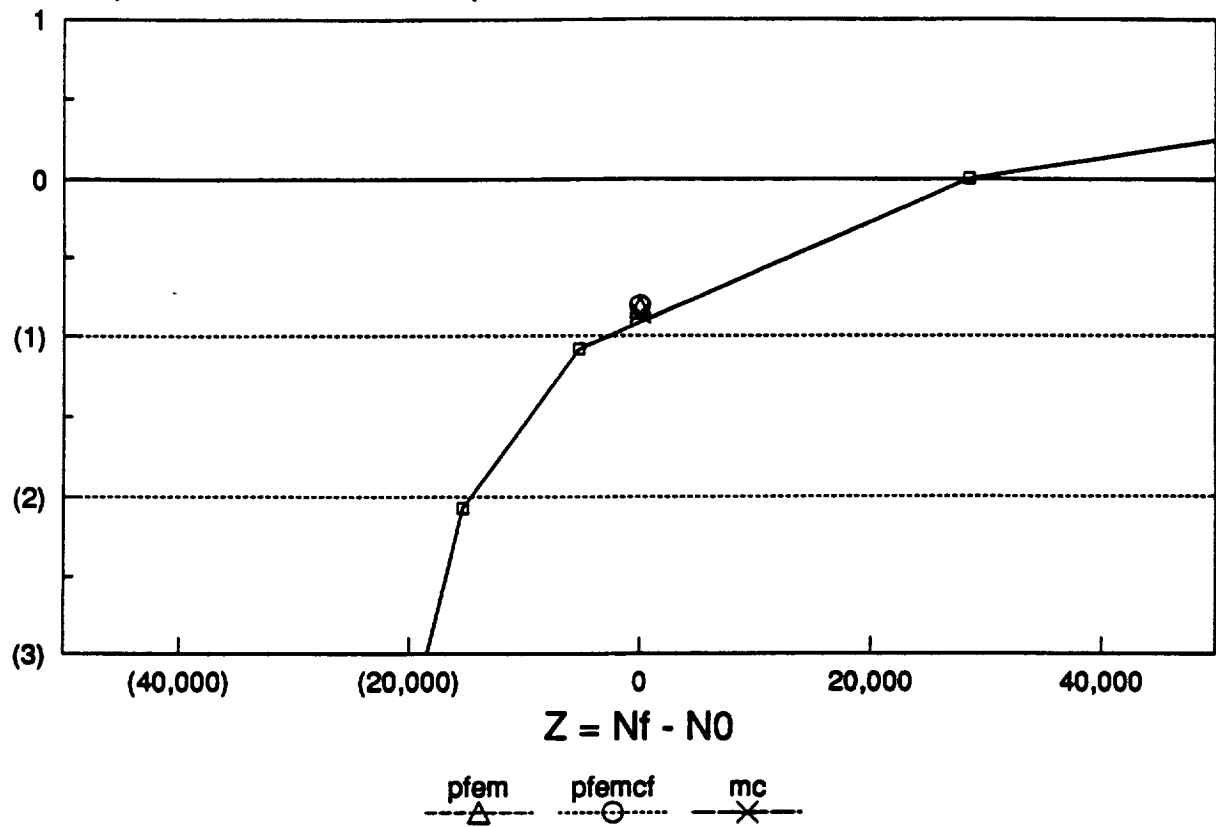


Figure 1-4 Probability of Failure Results for Three-Point Bend Specimen

Listing 1-1 NESSUS/PFEM Input Deck for the Three-Point Bend Demonstration

```

*PFEM
C PARIS FATIGUE CRACK PROPAGATION DEMONSTRATION (BHT 9/90)
C
C The problem is to compute the probability that NF is less than
C N0 where NF is the number of cycles to failure and N0 is the
C design life.
C
C The material is 2024-T3 Aluminum. A Paris Law is used to describe
C the crack growth with an assumed detectable flaw size.
C
C Random variables include:
C
C 1 PMAX point load acting at the center of the beam
C 2 S length of the beam
C 3 B height of the beam
C 4 AF final crack length
C 5 AO initial crack length
C 6 C Paris coefficient
C 7 Y stress intensity geometry factor
C 8 N0 design life
C
C The Paris exponent N is considered deterministic and the geometry
C factor is assumed constant for all crack lengths between AO and AF.
C
C The problem is solved several ways:
C
C (1) PFEM using FEM
C (2) PFEM using closed form equations
C (3) FPI using closed form equations
C (4) Monte Carlo using closed form equations
C
C The short subroutine PARISNF is used to compute the number of cycles
C to failure (Nf) from the input variables. This subroutine is called
C from both the UZFUNC (PFEM) and RESPON (FPI) routines.
C
C.....
C
C Z-FUNCTION DEFINITION
C
C THE "PARISNF" ROUTINE IS CODED AS USER RESISTANCE MODEL #3
C 8 RANDOM VARIABLES (3-COMPUTATIONAL,5-EXPLICIT)
C 1 USER COEFFICIENT (PARIS EXPONENT,N)
C
*ZFDEFINE
*COMPUTATIONALMETHOD 1 3
1 2 3
*EXPLICITVARIABLES 5
4 5 6 7 8
*ZFUNCTION 3 1
4.0

```

```

*UZFUNCTION
*END
C
C RANDOM VARIABLES
C
C ONE-HALF SYMMETRY IS USED FOR THE FEM MODEL; THEREFORE, THE LOADING
C AND BEAM LENGTH IS HALF OF THE ACTUAL VALUE.
C
*RVDEFINE
*DEFINE 1
  PMAX
  3.333333 0.333333 LOGNORMAL
  FORCES
  9 2 1.0
*DEFINE 2
  S
  4.0 0.4 NORMAL
  COORDINATES
  1 0.0 0.0
  2 0.5 0.0
  3 1.0 0.0
  4 0.0 0.0
  5 0.5 0.0
  6 1.0 0.0
  7 0.0 0.0
  8 0.5 0.0
  9 1.0 0.0
*DEFINE 3
  B
  2.0 0.2 NORMAL
  COORDINATES
  1 0.0 0.0
  2 0.0 0.0
  3 0.0 0.0
  4 0.0 0.5
  5 0.0 0.5
  6 0.0 0.5
  7 0.0 1.0
  8 0.0 1.0
  9 0.0 1.0
*DEFINE 4
  AF
  0.09 0.009 NORMAL
*DEFINE 5
  AO
  0.01 0.002 NORMAL
*DEFINE 6
  C
  1.16E-9 1.16E-10 NORMAL
*DEFINE 7
  Y
  1.0 0.1 NORMAL
*DEFINE 8
  NO

```

```

      20000.0  20.0  NORMAL
*PERT      1
      1      0.1
*PERT      2
      2      0.1
*PERT      3
      3      0.1
*PERT      4
      4      0.1
*PERT      5
      5      0.1
*PERT      6
      6      0.1
*PERT      7
      7      0.1
*PERT      8
      8      0.1
*END
C
C
C MEAN VALUE PROBABILISTIC ANALYSIS
C
*MVDEFINE
*PERT 8
1 2 3 4 5 6 7 8
*RANVAR 8
1 2 3 4 5 6 7 8
*DATATYPE 0
*RESPTYPE 3
*CONDITION 0
*NODE 3
*COMPONENT 1
*END
C
C
C ADVANCED MEAN VALUE PROBABILISTIC ANALYSIS
C
*AMVDEFINE
*ITERATION
  10 0.0025
*CONDITION 0
*NODE 3
*COMPONENT 1
*END
C
C END PFEM INPUT
C
*END
C
*FEM
C PARIS FATIGUE CRACK PROPAGATION DEMONSTRATION (BHT 9/90)
*CONS 0
*ELEMENTS 4
  151

```

```

*NODES 9
*BOUNDARY 4
*FORCES 1
*PRINT
*END
*ITER 0 3
20 0.01
*COORDINATES
1 0.0 0.0
2 2.0 0.0
3 4.0 0.0
4 0.0 1.0
5 2.0 1.0
6 4.0 1.0
7 0.0 2.0
8 2.0 2.0
9 4.0 2.0
*ELEMENTS 151
1 1 2 5 4
2 2 3 6 5
3 4 5 8 7
4 5 6 9 8
*BOUNDARY
1 2 0.0
3 1 0.0
6 1 0.0
9 1 0.0
*PROPERTY
1 9 1.0 30.0E3 0.3 1.0 1.0
*FORCES
9 2 3.333333
*PRINT
TOTAL NODE
STRESS NODE
*END
C
C FPI ANALYSIS CONTROL CARDS
C
*FPI
C PARIS FATIGUE CRACK PROPAGATION DEMONSTRATION (BHT 9/90)
*RVNUM 8
*GFUNCTION 1
*DATASETNM 9
*METHOD 1
*PRINTOPT 0
*ANALTYPE 1
*END
*ZLEVELS 1
0.0
*END

```

Listing 1-2 PARISNF Subroutine

```

SUBROUTINE PARISNF (NFMVR, FEMRES, PMAX, S, B, AF, AO, C, Y, RN0, N, ZVALUE)
IMPLICIT DOUBLE PRECISION (A-H, O-Z)

C
C FATIGUE CRACK PROPAGATION MODEL (PARIS LAW)
C
C .....
C
C NFMVR IS THE NUMBER OF FEM RESPONSE VARIABLES. IF >0, GET STRESS FROM
C FEMRES. IF NFMVR=0, STRESS IS COMPUTED USING CLOSE FORM EXPRESSION.
C NOTE THAT IN THIS CASE, THE LOAD (PMAX) AND BEAM LENGTH (S) ARE
C MULTIPLIED BY 2 SINCE HALF THE VALUES ARE USED IN THE INPUT DECK
C FOR THE SYMMETRY FEM MODEL.
C
      IF (NFMVR.GT.0) THEN
        SIGMAMAX = FEMRES
      ELSE
        TWOS      = S*2.0D0
        TWOP      = PMAX*2.0D0
        SIGMAMAX = 3.0D0*TWOP*TWOS/(2.0D0*B**2)
      ENDIF
      PI = ACOS(-1.0D0)

C
C COMPUTE THE NUMBER OF CYCLES TO FAILURE.
C
      RNF = 2*(AF**(N/2)*AO - AF*AO**(N/2))/
      +      (AF**(N/2)*AO**(N/2)*C*(-2 + N)*PI**(N/2)*SIGMAMAX**N*Y**N)

C
C FORMULATE THE Z-FUNCTION. NOTE THIS IS SIMILAR TO S-R SO THE FUNCTION
C WILL BE <= 0 WHEN FAILURE.
C
      ZVALUE = RNF - RN0

C
      RETURN
      END

```

Listing 1-3 Input Deck for the Monte Carlo Simulation

```

*FPI
C PARIS FATIGUE CYCLES TO FAILURE DEMONSTRATION
*RVNUM 8
*GFUNCTION 6
*METHOD 6
*PRINTOPT 0
*ANALTYPE 1
*END
*EXACTPRM
  18 1 0
  4.0
*MONTE
  100000, 1234.0, 1.0
*ZLEVELS 1
  0.0
*DEFNANVR
PMAX
  3.33333      0.333333      4.0
S
  4.0          0.4          2.0
B
  2.0          0.2          2.0
AF
  0.09         0.009        2.0
AO
  0.01         0.002        2.0
C
  0.116000E-08 0.116000E-09 2.0
Y
  1.0          0.1          2.0
NO
  20000.0      20.0         2.0
*END

```

1 Elastoplastic Low Cycle Fatigue Life of a Rectangular Plate

1.1 Introduction

The probabilistic life of a simple rectangular plate is analyzed considering damage caused by low cycle fatigue. In addition, a material strength degradation model is used in the analysis that considers the effect of thermal fatigue cycles and temperature on the initial yield stress.

During damage due to low cycle fatigue, significant amounts of plastic strain are induced each cycle. In this regime of plastic behavior, the fatigue life is much more accurately described as a function of the cyclic strain amplitude as opposed to the cyclic stress amplitude in the case of high cycle fatigue. Nevertheless, the effects of mean stress can be important, especially in the transition region between high and low cycle fatigue. For this demonstration, the number of cycles to failure is computed from,

$$N_f = A 10^{-\sigma_m} (\Delta \epsilon)^b$$

where N_f is the computed number of cycles to failure, A and b are material parameters, σ_m is the mean stress measure, and $\Delta \epsilon$ is the cyclic strain range measure. Here, these measures are computed as,

$$\sigma_m = \frac{1}{2} (\sigma_{\max} + \sigma_{\min})$$

$$\Delta \epsilon = \epsilon_{\max}^{pl} - \epsilon_{\min}^{pl}$$

where σ is the y component of the stress and ϵ^{pl} is the plastic strain.

1.2 Description of the Problem

Figure 1-1 illustrates the analysis performed. The rectangular plate is subjected to fully reversed strain cycling under uniaxial stress conditions. A bilinear stress-strain relationship is used to simplify the calculations for the demonstration. As shown in Figure 1-1, the imposed displacement cycles the plate through one and one-quarter cycles.

The deterministic elastoplastic calculations were checked by computing the stress and strain at six points, illustrated in Figure 1-5. Going point by point, the following relationships are easily seen,

$$\sigma_1 = \sigma_y$$

$$\epsilon_1 = \frac{\sigma_1}{E}$$

$$\epsilon_2 = \epsilon_{\max}$$

$$\sigma_2 = \sigma_1 + E'(\epsilon_2 - \epsilon_1)$$

$$\sigma_3 = -\sigma_2$$

$$\epsilon_3 = \epsilon_2 + \frac{(\sigma_3 - \sigma_2)}{E}$$

$$\epsilon_4 = \epsilon_{\min}$$

$$\sigma_4 = \sigma_3 + E'(\epsilon_4 - \epsilon_3)$$

$$\sigma_5 = -\sigma_4$$

$$\epsilon_5 = \epsilon_4 + \frac{(\sigma_5 - \sigma_4)}{E}$$

$$\epsilon_6 = \epsilon_{\max}$$

$$\sigma_6 = \sigma_5 + E'(\epsilon_6 - \epsilon_5)$$

where E is the elastic modulus, E' is the hardening slope taken from a uniaxial tension stress-strain curve, σ_y is the initial yield stress, ϵ_{\min} is the minimum total strain, and ϵ_{\max} is the maximum total strain. Because the displacement history is specified, the maximum and minimum stress occurs at points 6 and 4, respectively, and the maximum and minimum plastic strain occurs at points 2 and 4, respectively. Substituting the relations given above, the mean stress and plastic strain range are,

$$\sigma_m = \frac{(\sigma_6 + \sigma_4)}{2} = \frac{E'(2E' - E)(2EE'\epsilon_{\max} - E^2\epsilon_{\max} + E^2\epsilon_{\min} + 2E\sigma_y - 2E'\sigma_y)}{2E^3},$$

$$\Delta\epsilon^p = \frac{2E\sigma_m}{E'(E - 2E')}.$$

1.3 Modeling and Methods

A finite element model of the rectangular plate was constructed using two square plane stress linear elements. The boundary conditions and loading on the plate are shown in Figure 1-1. The applied displacement history is shown in Figure 1-2.

Nonlinear material behavior is allowed using the bilinear stress-strain relation shown in Figure 1-3. The classical von Mises J_2 -flow plasticity model is used with isotropic hardening. For the finite element calculation of mean stress and plastic strain range, the inelastic portion of the stress-strain relationship is described in terms of the effective plastic strain (as opposed to the effective strain). In the case of linear hardening, a single parameter H (the hardening slope) is input, computed by,

$$H = \frac{EE'}{E - E'}.$$

The material damage model used in this demonstration is the multi-factor interaction (MFI) relation of Boyce [1]. Although, in general, a more realistic damage model may be warranted, the MFI model provides a simple mechanism for treating any possible material strength damage indicator, such as crack length, fatigue strength, or stiffness. The MFI relation models material degradation using a series product equation fitted to experimental or assumed data. A single term is used to describe each primitive variable (stress, temperature, cycles, etc.). The general form of the MFI relation is,

$$\frac{M_p}{M_{p_0}} = \prod_{i=1}^n \left(\frac{A_{iF} - A_i}{A_{iF} - A_{i0}} \right)^{a_i}$$

where A_{iF} , A_i , and A_{i0} are the ultimate, current, and reference values of the i^{th} primitive variable, a_i is the value of an empirical constant for the i^{th} primitive variable, n is the number of primitive variables considered, and M_p and M_{p_0} are the current and reference values of the material property. The effect of each primitive variable on the current material property varies from 1.0 (no effect on strength) to 0.0 (complete loss of strength) times the reference value of the material property.

Degradation of the initial yield stress is modeled as a function of temperature and thermal fatigue primitive variables. Consistent with the linear hardening assumption, the current value of the yield stress is computed by,

$$\sigma_y = \sigma_y^i + H \bar{\epsilon}^{pl},$$

$$\sigma_y^i = \sigma_y^0 \left(\frac{N_{TF} - N_T}{N_{TF} - N_{T0}} \right)^q \left(\frac{T_F - T}{T_F - T_0} \right)^u.$$

where σ_y^i is the damaged initial yield stress, H is the hardening slope, $\bar{\epsilon}^{pl}$ is the equivalent plastic strain, and σ_y^0 is the yield strength of the undamaged material. N_{TF} , N_T , and N_{T0} are the final, current, and initial number of thermal fatigue cycles, and T_F , T , and T_0 are the final, current, and initial temperatures.

The deterministic finite element calculations were checked against the analytical solution, and are shown in Table 1-1. Since a relatively loose convergence tolerance (1%) was used, the agreement shown in Table 1-1 is acceptable. The y-component stress-strain response at node 5 is plotted in Figure 1-4.

Table 1-1. Comparison of Analytical and Finite Element Solutions. ϵ_y is the y-component of the total strain (applied), σ_y is the y-component of the Stress and ϵ_y^p is the y-component of the Plastic Strain.

		Analytical		NESSUS/FEM	
Point	ϵ_y	σ_y	ϵ_y^p	σ_y	ϵ_y^p
1	0.003471	0.243	0.0		
2	0.015	0.2968	0.011529	0.2975	0.01075
3	0.00641	-0.2968	0.011529		
4	-0.015	-0.39723	-0.009991	-0.3977	-0.009322
5	-0.003352	0.39723	-0.009991		
6	0.015	0.48426	0.008659		

Seven random variables, listed in Table 1-2, were considered in the analysis. The values for the terms in the MFI relation were calculated from a regression analysis of cast nickel-based superalloy data, computed and tabulated in [2]. The values for the remaining primitive random variables were arbitrarily chosen as:

$$N_{TF} = 1000 \text{ cycles};$$

$$N_{T0} = 10 \text{ cycles};$$

$$T_F = 2732^\circ \text{ F};$$

$$T_0 = 68^\circ \text{ F};$$

$$b = -4.$$

Table 1-2. Random Variables Used in the Low Cycle Fatigue Demonstration Problem

Variable	Description	Mean	Standard Deviation	Coefficient of Variation	Distribution
σ_y^0	Material Yield Strength	0.5165	0.02583	5%	Normal
N_T	Number of Thermal Fatigue Cycles	501	50.1	10%	Lognormal
q	Thermal Fatigue Damage Exponent	0.5	0.015	3%	Normal
T	Temperature	1562	46.7	3%	Normal
u	Thermal Damage Exponent	0.5	0.015	3%	Normal
H	Hardening Slope	5.0	0.25	5%	Normal
A	LCF Material Parameter	$6.304(10)^{-3}$	$3.152(10)^{-4}$	5%	Normal

NESSUS Finite Element Analysis

The input deck for the NESSUS/PFEM analysis is shown in Listing 1-1. The calculation of N_f is coded in the UZFUNC subroutine, shown in Listing 1-2, and selected as model 11 on the *ZFUNCTION keyword card. The material exponent, b , which is considered deterministic, is passed into UZFUNC as a user coefficient, also on the *ZFUNCTION card.

Four computational variables (CVARIABLE) are defined: (1) the maximum plastic strain, (2) the minimum plastic strain, (3) the maximum stress, and (4) the minimum stress. These four variables are passed into the UZFUNC user-routine (for the deterministic and each perturbed solution) where the mean stress and strain range are computed and used in the calculation of N_f .

The seven random variables considered in the analysis are defined next in the *RVDEFINE section. Random variables 1-6 (σ_y^0 , N_T , q , T , u , and H) are defined as COEF (coefficient) random variables. The COEF random variables are perturbed and tracked during the course of the finite element calculations in the same fashion as other FEM random variables. In this demonstration, the COEF variables are used to define the yield stress damage model in the UWKSL user-routine, given in Listing 1-3. The seventh random variable, the material parameter A , is an explicit random variable; therefore, the type and data block are not input.

Figure 1-5 diagrams the effect of a perturbation of the initial yield stress which results from a perturbation of any of the primitive random variables σ_y^0 , N_T , q , T , or u . Figure 1-6 shows the effect of a perturbation of primitive random variable H .

The Mean-Value and Advanced Mean Value Iteration analysis options are specified in the MVDEFINE and AMVDEFINE sections, respectively. To summarize, all seven random variables and perturbations are selected, and the Advanced Mean Value iteration procedure is set to continue until convergence is within 5% of the computed life or a maximum of 5 iterations is reached.

To verify the demonstration, the analytical solution for the analysis was programmed in the RESPON routine and a Monte Carlo simulation performed. The Monte Carlo input is given in Listing 1-4, and the RESPON subroutine given in Listing 1-5.

1.4 Results and Discussion

The cumulative distribution function (CDF) for the number of cycles to failure is given in Figure 1-8. As shown, the agreement between the NESSUS AMV+ procedure and the Monte Carlo simulation is quite good. Note that with the NESSUS AMV+ procedure, 52 response function evaluations (NESSUS/FEM re-solutions, including perturbed solutions) were required as opposed to 100,000 evaluations using Monte Carlo.

The probabilistic sensitivity factors for each primitive random variable is plotted in Figure 1-9 as a function of probability (CDF value). As shown, N_T declines in importance and σ_y^0 increases in importance of as one moves from lower probability to higher probability.

1.5 References

- [1] Boyce, L. and Chamis, C.C., "Probabilistic Constitutive Relationships for Cyclic Material Strength Models, " *Proc. 29th Structures, Structural Dynamics and Materials Conf.*, Williamsburg, VA, April, 1988.
- [2] Boyce, L. and Chamis, C.C., "Probabilistic Constitutive Relationships for Material Strength Degradation Models, " *Proc. 30th Structures, Structural Dynamics and Materials Conf.*, Mobile, AL, April, 1989.

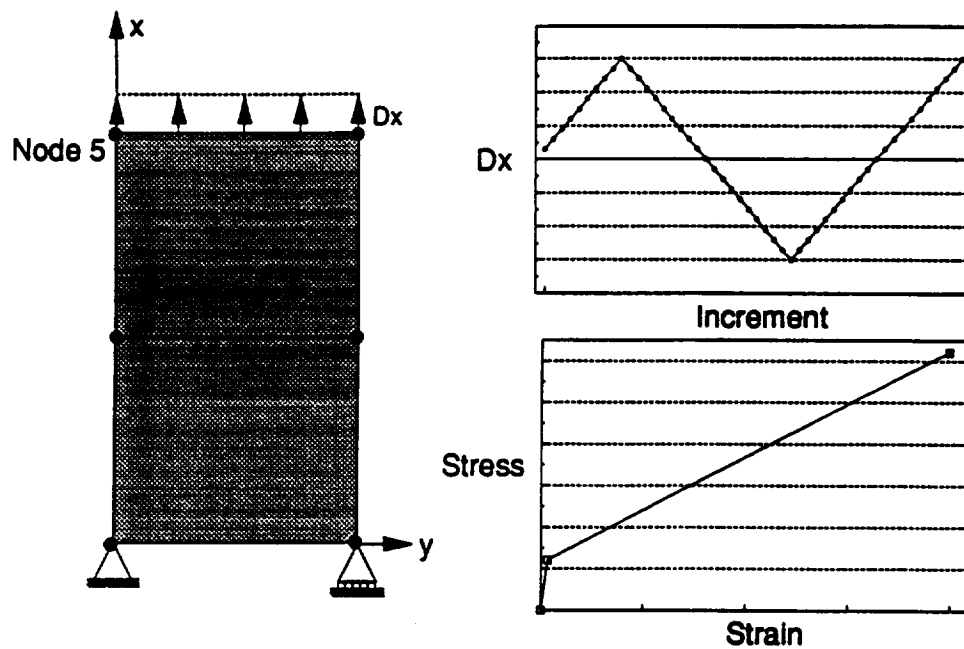


Figure 1-1 Elastoplastic Rectangular Plate Analysis

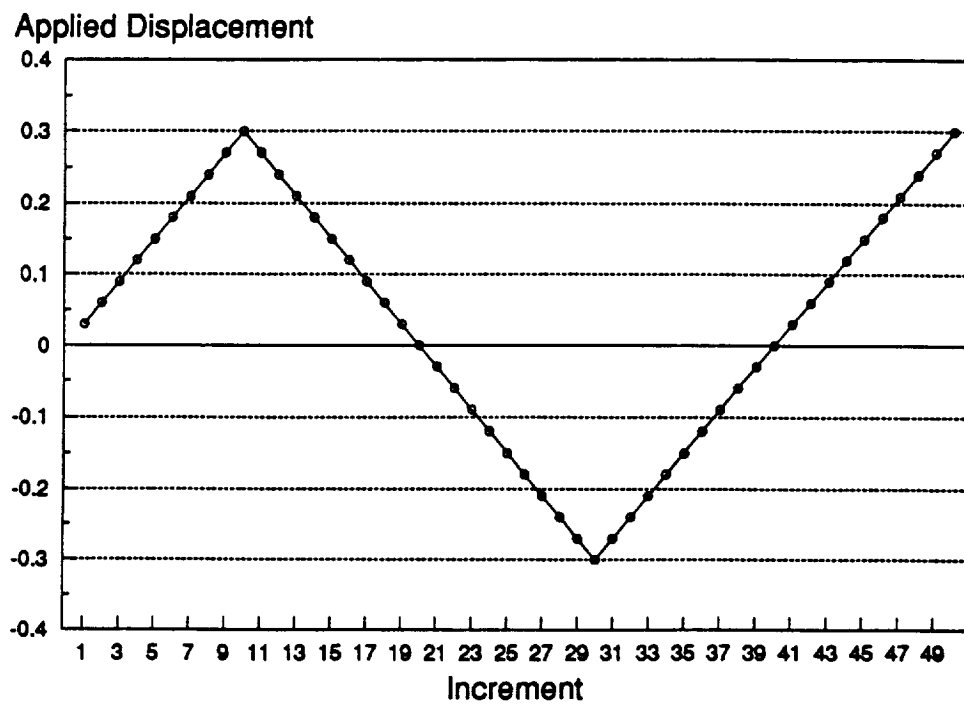


Figure 1-2 Displacement Loading History

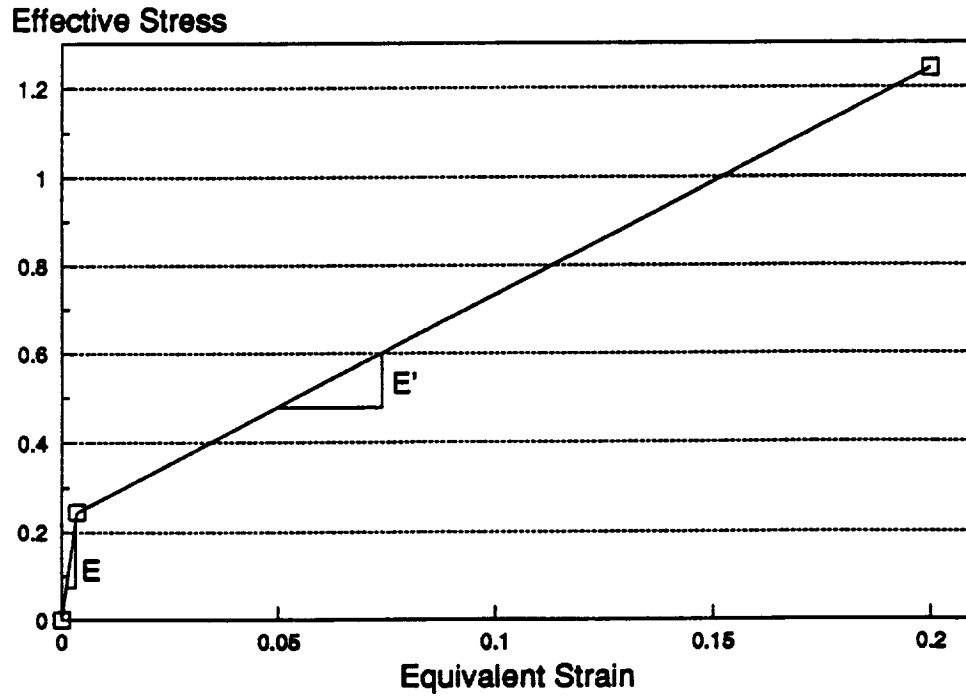


Figure 1-3 Material Stress-Strain Curve

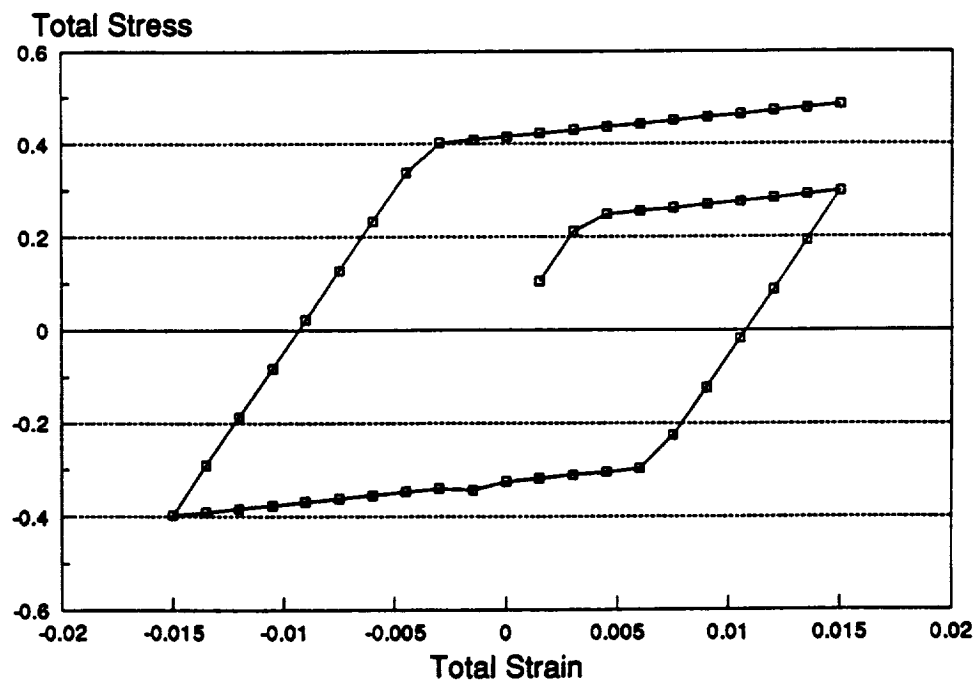


Figure 1-4 Stress-Strain Response at Node 5, y-component

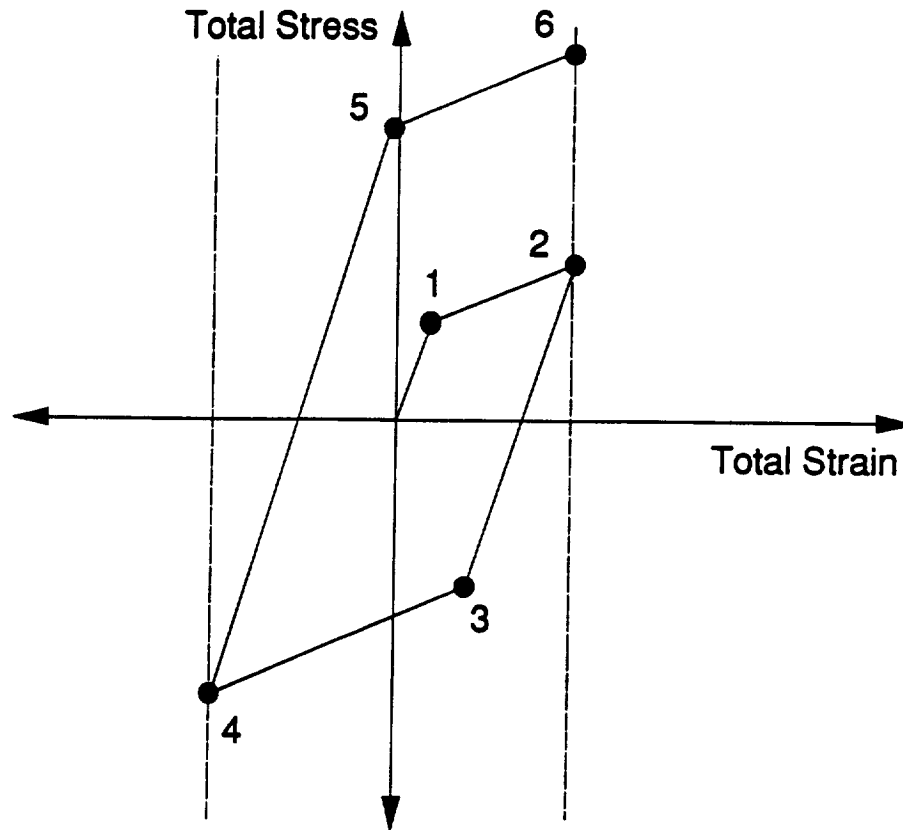


Figure 1-5 Hysteresis Loop Diagram Showing Analytical Calculation Points

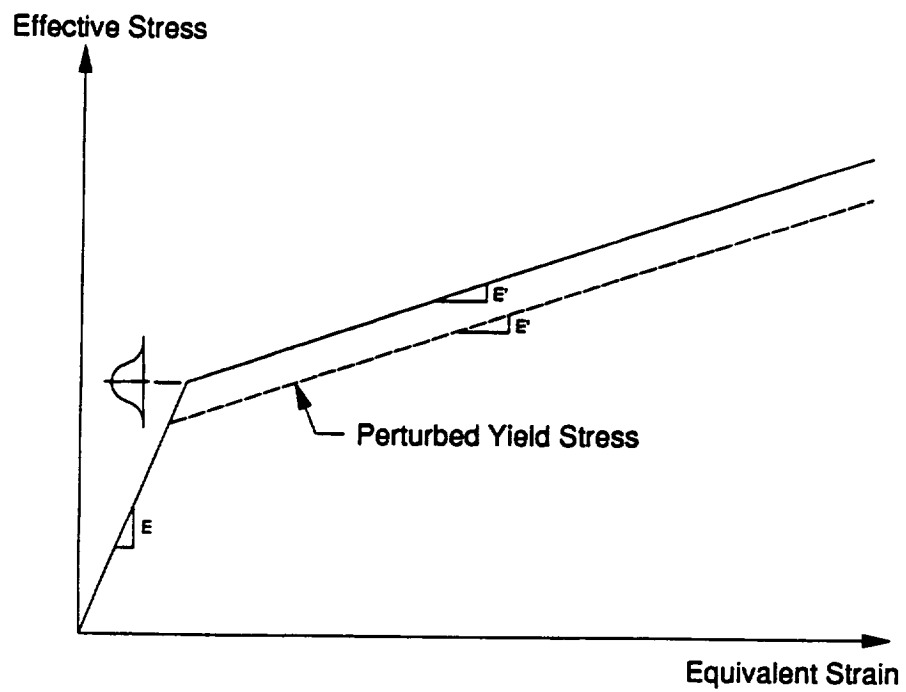


Figure 1-6 Perturbation in Yield Stress σ_y

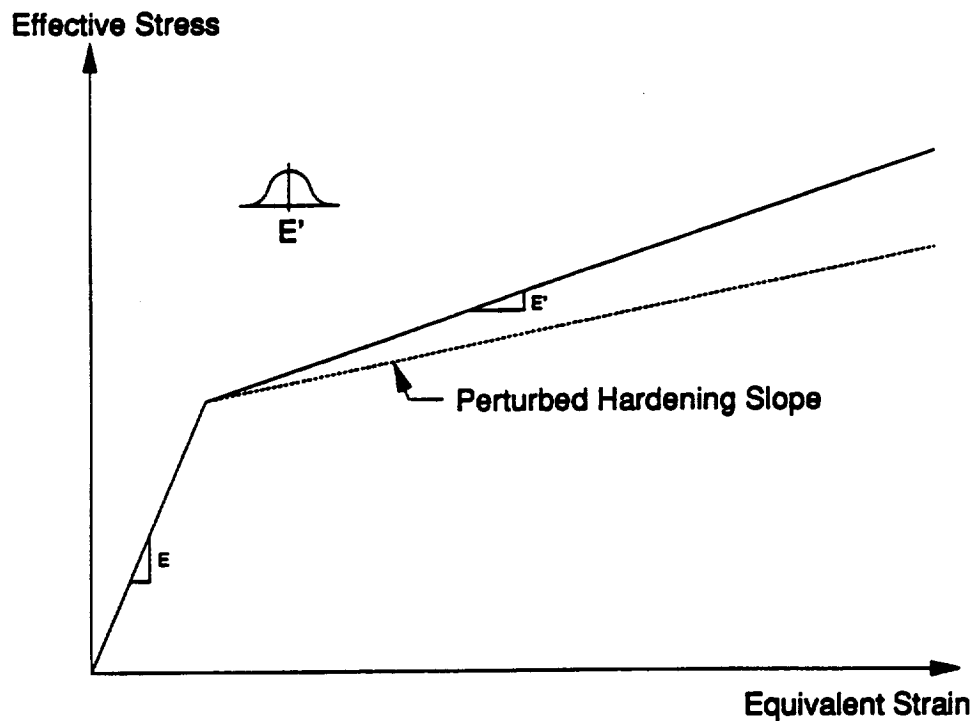


Figure 1-7 Perturbation in Hardening Slope E'

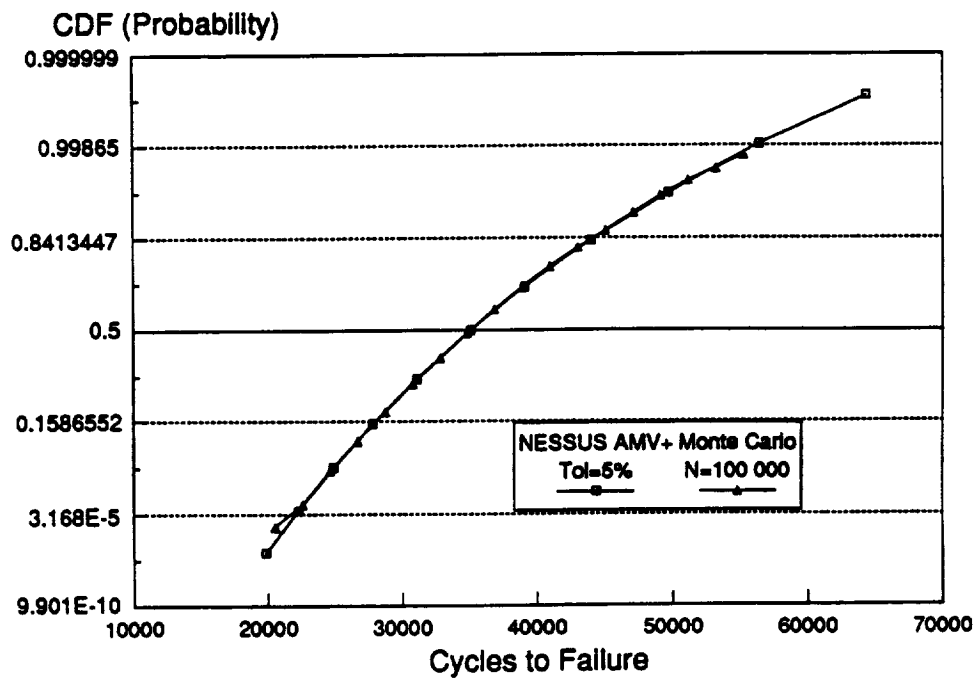


Figure 1-8 Cumulative Distribution Function of the Cycles to Failure Computed by NESSUS AMV+ Procedure and Monte Carlo Simulation.

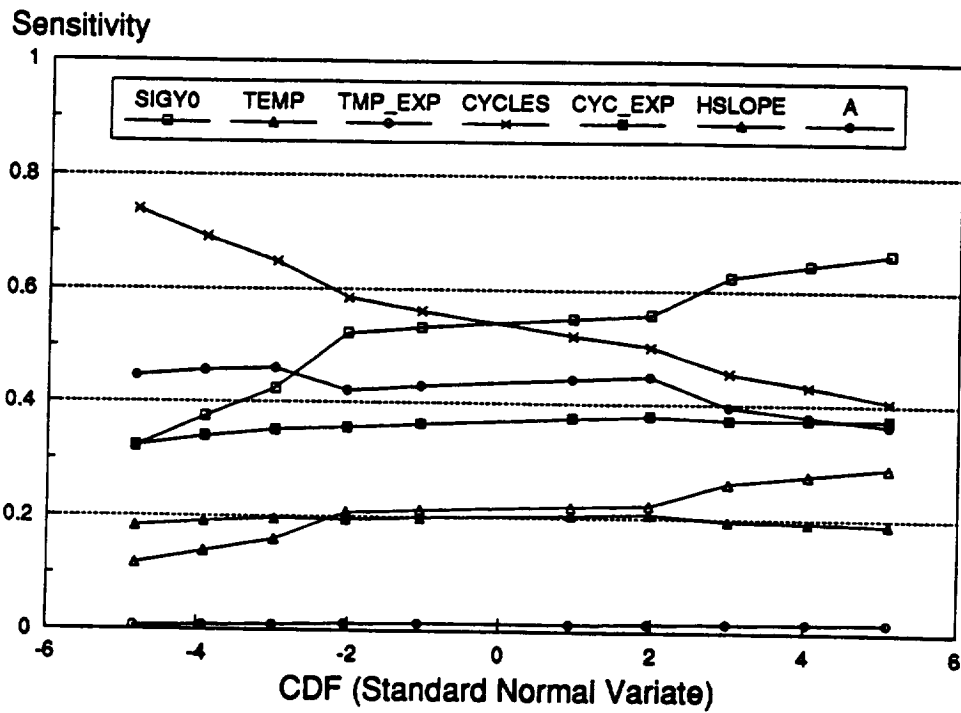


Figure 1-9 Primitive Random Variable Probabilistic Sensitivity Factors Plotted as a Function of CDF.

Listing 1-1 NESSUS/PFEM Input Deck

```

*PFEM
C ... LOW CYCLE/MFI FATIGUE DEMONSTRATION (MFI YIELD STRESS DEGRADATION)
C
*ZFDEFINE
  *COMPUTATIONALMETHOD 1 6
  1 2 3 4 5 6
  *EXPLICITVARIABLES 1
  7
  *ZFUNCTION 11 1
  4.0
  *UZFUNCTION
C
C C VARIABLE 1 ... ALGEBRAIC MAXIMUM PLASTIC STRAIN AT NODE 5
C
  *C VARIABLE 1
  RESPTYPE 11
  COMPONENTLIST 1
  1
  NODELIST 1
  5
  CONDITIONLIST 50
  0 1 2 3 4 5 6 7 8 9
  10 11 12 13 14 15 16 17 18 19
  20 21 22 23 24 25 26 27 28 29
  30 31 32 33 34 35 36 37 38 39
  40 41 42 43 44 45 46 47 48 49
  OPERATION 1
  END
C
C C VARIABLE 2 ... ALGEBRAIC MINIMUM PLASTIC STRAIN AT NODE 5
C
  *C VARIABLE 2
  RESPTYPE 11
  COMPONENTLIST 1
  1
  NODELIST 1
  5
  CONDITIONLIST 50
  0 1 2 3 4 5 6 7 8 9
  10 11 12 13 14 15 16 17 18 19
  20 21 22 23 24 25 26 27 28 29
  30 31 32 33 34 35 36 37 38 39
  40 41 42 43 44 45 46 47 48 49
  OPERATION 2
  END
C
C C VARIABLE 3 ... ALGEBRAIC MAXIMUM AXIAL STRESS AT NODE 5
C
  *C VARIABLE 3
  RESPTYPE 3
  COMPONENTLIST 1
  1
  NODELIST 1
  5
  CONDITIONLIST 50
  0 1 2 3 4 5 6 7 8 9
  10 11 12 13 14 15 16 17 18 19
  20 21 22 23 24 25 26 27 28 29
  30 31 32 33 34 35 36 37 38 39
  40 41 42 43 44 45 46 47 48 49
  OPERATION 1
  END

```

```

C
C C V A R I A B L E 4 ... A L G E B R A I C M I N I M U M A X I A L S T R E S S A T N O D E 5
C
  * C V A R I A B L E 4
    R E S P T Y P E 3
    C O M P O N E N T L I S T 1
      1
    N O D E L I S T 1
      5
    C O N D I T I O N L I S T 50
      0 1 2 3 4 5 6 7 8 9
      10 11 12 13 14 15 16 17 18 19
      20 21 22 23 24 25 26 27 28 29
      30 31 32 33 34 35 36 37 38 39
      40 41 42 43 44 45 46 47 48 49
    O P E R A T I O N 2
    E N D
  * E N D
C
C ..... R A N D O M V A R I A B L E S
C
  * R V D E F I N E
  * D E F I N E 1
    S I G Y 0
      0.516473 0.025824 N O R M A L
    C O E F
      1 1.0000
      2 0.0000
      3 0.0000
      4 0.0000
      5 0.0000
      6 0.0000
  * D E F I N E 2
    T E M P
      1562.0 46.7 N O R M A L
    C O E F
      1 0.0000
      2 1.0000
      3 0.0000
      4 0.0000
      5 0.0000
      6 0.0000
  * D E F I N E 3
    T M P _ E X P
      0.5 0.0015 N O R M A L
    C O E F
      1 0.0000
      2 0.0000
      3 1.0000
      4 0.0000
      5 0.0000
      6 0.0000
  * D E F I N E 4
    C Y C L E S
      501.0 50.1 L O G N O R M A L
    C O E F
      1 0.0000
      2 0.0000
      3 0.0000
      4 1.0000
      5 0.0000
      6 0.0000
  * D E F I N E 5
    C Y C _ E X P

```

```

0.5      0.05      NORMAL
COEF
  1      0.0000
  2      0.0000
  3      0.0000
  4      0.0000
  5      1.0000
  6      0.0000
*DEFINE   6
HSLOPE
  5.0      0.25      NORMAL
COEF
  1      0.0000
  2      0.0000
  3      0.0000
  4      0.0000
  5      0.0000
  6      1.0000
*DEFINE   7
A
6.30351E-3  3.15176E-4  NORMAL
C
C ..... PERTURBATIONS
C
*PERT    1
  1      0.1000
*PERT    2
  2      0.1000
*PERT    3
  3      0.1000
*PERT    4
  4      0.1000
*PERT    5
  5      0.1000
*PERT    6
  6      0.1000
*PERT    7
  7      0.1000
*END
C
C ..... MEAN VALUE PROBABILISTIC ANALYSIS
C
*MVDEFINE
  *PERTURBATIONS 7
    1 2 3 4 5 6 7
  *RANVAR 7
    1 2 3 4 5 6 7
*END
C
C ..... ADVANCED MEAN VALUE PROBABILISTIC ANALYSIS
C
*AMVDEFINE
  *NODE 1
  *COMPONENT 1
  *CONDITION 1
  *ITERATION
    5 0.05
*END
*END
C
*FEM
C ... LOW CYCLE/MFI FATIGUE DEMONSTRATION (MFI YIELD STRESS DEGRADATION)
C
C TWO ELEMENT TEST OF THE VON MISES PLASTICITY MODEL WITH ISOTROPIC

```

C HARDENING. A BI-LINEAR PLASTIC STRAIN CURVE IS USED AND IS DEFINED IN
C THE USER SUBROUTINE "UWKSL". THE YIELD STRESS (SIGY) IS DEFINED USING
C THE MULTI-FACTOR INTERACTION (MFI) RELATION (BOYCE, 1988). THE MFI
C MODEL IS USED TO DESCRIBE THE INITIAL YIELD STRESS AS A FUNCTION OF
C RANDOM (COEF) PRIMITIVE VARIABLES.

C
C 23-NOV-90
C

C.....

C
C ... PARAMETER DATA
C

*TANGENT 1
*CONSTITUTIVE 2
*NODES 6
*ELEMENTS 2
3
*BOUNDARY 5
*HARDENING 2
*UWKSL
*COEF 6
*PRINT
*MONITOR 4
*END

C
C ... MODEL DATA AND LOAD INCREMENT 0
C

*ITERATIONS 0
20 0.0100
*ITERATIONS 1 6
20 0.0000 0.0001
*ELEMENTS 3
1 1 3 4 2
2 3 5 6 4
*COORDINATES
1 0.0000 0.0000
2 0.0000 10.0000
3 10.0000 0.0000
4 10.0000 10.0000
5 20.0000 0.0000
6 20.0000 10.0000
*BOUNDARY
1 1 0.0000
1 2 0.0000
2 1 0.0000
5 1 0.0300
6 1 0.0300
*PROPERTY 3
1 6 1.0000 70.0000 0.2000
*PRINT
TOTA NODE
STRE NODE
STRA NODE
PLAS NODE
*COEF
1 0.516473
2 1562.0
3 0.5
4 501.0
5 0.5
6 5.0
*MONITOR
TOTA NODE 5 COMP 1
STRE NODE 5 COMP 1

```

    STRA NODE 5 COMP 1
    PLAS NODE 5 COMP 1
*END
C
C ... CONTINUE LOADING FOR NINE MORE INCREMENTS
C
*AUTO
  9
*END
C
C ... UNLOAD FOR THE NEXT TWENTY INCREMENTS
C
*BOUN
  1      1      0.0000
  1      2      0.0000
  2      1      0.0000
  5      1     -0.0300
  6      1     -0.0300
*AUTO
  20
*END
C
C ... RELOAD FOR THE LAST TWENTY INCREMENTS
C
*BOUN
  1      1      0.0000
  1      2      0.0000
  2      1      0.0000
  5      1      0.0300
  6      1      0.0300
*AUTO
  20
*END
*EXIT
C
C ..... FPI INPUT DATA
C
*FPI
C ... LOW CYCLE/MFI FATIGUE DEMONSTRATION (MFI YIELD STRESS DEGRADATION)
*RVNUM 7
*DATASETS 8
*GFUNCTION 1
*METHOD 1
*ANALYTYPE 0
*PRINT 0
*END
*END

```

Listing 1-2 NESSUS/FEM UWKSL User Routine used to Define the Work Hardening Curve as
a Function of the Equivalent Plastic Strain and the MFI Model

```

C ... SUBROUTINE UWKSL ... USER-DEFINED WORKHARDENING SLOPE
C
C   SUBROUTINE UWKSL
C     1 (EPLAS ,SIGY ,HSLOP ,SHIFT ,TEMP ,NODE )
C
C *****
C **
C **   DEFINES THE WORKHARDENING SLOPE FOR J2-FLOW PLASTICITY
C **
C *****
C
C ARGUMENTS:
C
C   EPLAS   INPUT   THE EQUIVALENT PLASTIC STRAIN VALUE
C   SIGY    OUTPUT  THE YIELD STRESS AT THIS PLASTIC STRAIN
C   HSLOP   OUTPUT  THE SLOPE OF THE WORKHARDENING CURVE
C   SHIFT   OUTPUT  THE SHIFT TENSOR (BACKSTRESS) VALUE
C   TEMP    INPUT   THE TEMPERATURE AT THE NODE
C   NODE    INPUT   THE NODE NUMBER
C
C NOTES:
C
C   * THIS SUBROUTINE IS CALLED BY:
C
C     YIEL      TO CHECK WETHER PLASTIC YIELDING OCCURS
C
C   * THE VALUE OF THE EQUIVALENT PLASTIC STRAIN IS A GIVEN QUANTITY.
C     IT WILL BE USED TO INTERPOLATE THE TABLES CODED BY THE USER IN
C     THIS SUBROUTINE OR AS AN INPUT FOR A FUNCTIONAL EXPRESSION.
C
C   * THE YIELD STRESS AT THIS PLASTIC STRAIN IS USED TO DETERMINE THE
C     RADIUS OF THE VON MISES YIELD SURFACE.
C
C   * THE SLOPE OF THE WORKHARDENING CURVE IS GIVEN BY
C
C       d Sigma
C
C     HSLOP = -----
C              pl
C       d Eps
C
C       WHERE d Sigma IS THE STRESS INCREMENT, AND d Epspl IS THE
C       CORRESPONDING PLASTIC STRAIN INCREMENT.
C
C   * THE SHIFT TENSOR (BACKSTRESS) VALUE IS THE AMOUNT SUBTRACTED
C     FROM THE EQUIVALENT STRESS BEFORE COMPUTING THE GEOMETRY OF
C     THE YIELD SURFACE; IN OTHER WORDS, IT IS THE AMOUNT BY WHICH
C     THE YIELD SURFACE IS SHIFTED FROM THE ORIGIN DURING KINEMATIC
C     OR COMBINED HARDENING; IT IS ZERO FOR ISOTROPIC HARDENING.
C
C   * THE TABLES AND/OR FUNCTIONS CODED IN THIS USER SUBROUTINE MAY
C     DEPEND ON BOTH SPATIAL LOCATION (NODE) AND THE TEMPERATURE.
C
C *****
C
C   IMPLICIT REAL*8 (A-H,O-Z)
C
C *****
C ** BLOCK USRCOF: USER-DEFINED COEFFICIENTS FROM "**COEF" OPTION **

```

```

C *****
C
C IUNIT <=> UNIT NUMBER OF MAIN OUTPUT FILE (FOR ANY USER-DEFINED
C OUTPUT THAT MAY BE GENERATED)
C JPert <=> CURRENT PERTURBATION NUMBER
C (0 .LE. JPert .LE. NUMBER OF PERTURBATIONS)
C NCOEF <=> NUMBER OF USER-DEFINED COEFFICIENTS (UP TO 10 ALLOWED)
C VCOEF <=> VALUES OF THE USER-DEFINED COEFFICIENTS FOR
C PERTURBATION # JPert
C
C COMMON / USRCOF / IUNIT ,JPert ,NCOEF ,VCOEF(10)
C
C .....
C
C MULTI-FACTOR INTERACTION RELATION USED TO DEFINE THE YIELD STRESS AS
C A RANDOM FUNCTION OF TEMPERATURE AND THERMAL FATIGUE CYCLES.
C
C SIGY0 = VCOEF(1)
C TT = VCOEF(2)
C QQQ = VCOEF(3)
C RNT = VCOEF(4)
C UUU = VCOEF(5)
C HSLOPE = VCOEF(6)
C
C HSLOP = HSLOPE
C SHIFT = 0.0D0
C TTF = 2732.0D0
C TT0 = 68.0D0
C RNTF = 1000.0D0
C RNT0 = 10.0D0
C
C SIGYI = SIGY0*
C + (( TTF - TT )/( TTF - TT0 ))**QQQ*
C + (( RNTF - RNT )/( RNTF - RNT0 ))**UUU
C SIGY = EPLAS*HSLOP + SIGYI
C WRITE(IUNIT,1) (VCOEF(I),I=1,5),SIGYI,EPLAS,SIGY
C 1 FORMAT(' <=UWKS> COEFS 1-5 = ',5(2X,E11.4),
C + ' SIGYI=',E11.4,' EPLAS=',E11.4,' SIGY=',E11.4)
C
C RETURN
C END

```

Listing 1-3 Section of Code from the UZFUNC User Routine used to Compute the Cycles to Failure from the NESSUS/FEM Analysis

```

C *****
C * MODEL 11: LCF/MFI DEMONSTRATION--2 ELEMENT ELASTIC PLATE TENSION
C * TEST WITH YIELD STRESS DEGRADED BY MFI RELATION.
C *****
C
1100 CONTINUE
      STNMAX = FEMRES(1)
      STNMIN = FEMRES(2)
      STRMAX = FEMRES(3)
      STRMIN = FEMRES(4)
      A = VALIV(7)
      B = INT(PFCOEF(1))
      DSTRN = STNMAX - STNMIN
      DSTRS = (STRMAX + STRMIN)/2.0
      VALDV = A/(10**DSTRS*DSTRN**B)
      WRITE(ICONSL,1101) DSTRN,DSTRS,VALDV
1101 FORMAT(' <=UZFUN=> DSTRN,DSTRS,NF = ',3(2X,E12.5))
      GOTO 9999

```

Listing 1-4 Input Deck for the Monte Carlo Simulation

```

*FPI
C ... LOW CYCLE/MFI FATIGUE DEMONSTRATION (MFI YIELD STRESS DEGRADATION)
*RVNUM 7
*GFUNCTION 6
*METHOD 6
*PRINTOPT 0
*ANALTYF 0
*END
*EXACTPRM
24,3,0
4.0
0.99867
0.93271
*MONTE
100000, 12345., 1.0
*DEFNANVR
SIGY0
0.516473 0.02582 2.0
TEMP
1562.0 46.7 2.0
TMP_EXP
0.5 0.0015 2.0
CYCLES
501.0 50.1 4.0
CYC_EXP
0.5 0.05 2.0
HSLOPE
4.66667 0.233333 2.0
A
6.30351E-3 3.15176E-4 2.0
*END

```

Listing 1-5 Section of Code from the RESPON User Routine used to Compute the Cycles to Failure from the NESSUS Monte Carlo Simulation

```

C
C*****
C* LOW CYCLE FATIGUE RESPONSE WITH ELASTOPLASTIC MATERIAL DAMAGE (MFI) *
C*****
C
C B.H.THACKER.....24-NOV-90
C
C 240 CONTINUE
C
C ...INITIALIZE
C
C      EMAX = 0.015
C      EMIN = -0.015
C      E = 70.0
C
C ...CURRENT VALUES FOR THE RANDOM VARIABLES
C
C      SIGY0 = XSTAR(1)
C      TT = XSTAR(2)
C      QQQ = XSTAR(3)
C      RNT = XSTAR(4)
C      UUU = XSTAR(5)
C      H = XSTAR(6)
C      A = XSTAR(7)
C      IB = INT(COEF(1))
C      SSFACT = COEF(2)
C      SNFACT = COEF(3)
C
C ...COMPUTE CURRENT YIELD STRESS USING MFI DAMAGE MODEL
C
C      TTF = 2732.0D0
C      TT0 = 68.0D0
C      RNTF = 1000.0D0
C      RNT0 = 10.0D0
C      SY = SIGY0*(( TTF - TT )/( TTF - TT0 ))**QQQ*
C      +      (( RNTF - RNT )/( RNTF - RNT0 ))**UUU
C
C ...COMPUTE MEAN STRESS AND PLASTIC STRAIN RANGE (CLOSED FORM)
C
C      DSTRS = H*(-E + 2*H)*(-(E**2*EMAX) + 2*E*H*EMAX +
C      +      E**2*EMIN + 2*E*SY - 2*H*SY)/(2*E**3)
C      DPSTRN = EMAX-EMIN+(-SY-H*(EMAX-SY/E)-(SY+H*(EMAX-SY/E)))/E
C
C ...ADJUST BY THE CORRECTION FACTORS (TO CHECK AGAINST NESSUS)
C
C      DSTRS = DSTRS*SSFACT
C      DPSTRN = DPSTRN*SNFACT
C
C ...AND FINALLY, THE NUMBER OF CYCLES TO FAILURE
C
C      RESPON = A/(10**DSTRS*DPSTRN**IB)
C      GOTO 9999

```


APPENDIX D

SSME Risk Research

Rocketdyne Division, Rockwell International

1 RISK DEFINITION AND MEASURE

Risk is defined as consequence of unreliability. Risk analysis involves three questions

1. What can happen?
2. What is the likelihood it will happen?
3. If it does happen, what are the consequences?

In some instances failure itself may be considered as a measure of risk. But, or generally risk is measured mostly in terms of consequences such as cost in dollars and program delay. It is possible to translate all the consequences to dollar cost using appropriate models. Risk can be portrayed as a curve of relative frequency vs risk measure (Figure 1). Such curves can then be used to determine acceptable risk. This form of portraying risk is useful, as the question of acceptable risk cannot be determined without considering cost and benefit.

In the space propulsion system risk analysis the entire range of risk curve is of interest (low consequence high frequency range, medium consequence medium consequence range, high consequence low frequency range, see Figure 1). The high consequence low frequency range deals with criticality one failures. The mid range deals with many maintenance items. The low range might be of importance when public good will and confidence in the program are to be considered.

2 EXPOSURE CLASSES

In the first step in risk calculation, underlying reliability of the components/systems has to be plotted as probability of failure vs exposure classes. The exposure classes deal with usage of the engine, and different failure modes will have different exposure items as relevant items. It is recommended for space propulsion systems that provisions be made for the following exposure classes. They are

1. Number of start/stops of the engine (LCF damage).
2. Total time in seconds as a function of power level (HCF damage).
3. Time at specific speed ranges (Resonance issues).

3 COST MODELS

In order to evaluate the consequence of engine loss, engine damage, and vehicle loss, it is necessary to have an algorithm to calculate the cost elements that contribute to it. For the purposes of PSAM applications, the aim is to develop a simple yet cost-effect model.

Considering engine cost, and considering top down, the engine cost is driven by major component costs. It consists of fabrication, support, and production material costs. For example, a typical propulsion system major component classification in terms of cost elements can be

1. Engine Final Assembly.
2. Low Pressure Pumps.
3. High Pressure Pumps.
4. Main Combustion Chamber.
5. Power Head.
6. Ducts.
7. System Installation.
8. Electrical Harness.

9. Actuation Systems.

Given the fabrication/assembly man-hours, a composite wrap rate, and production material costs, the total cost of the engine can be calculated. Additional items to be considered include lump sum items which are supplied by subcontractors. If one is only interested in catastrophic failures, then the engine cost can be added to other cost element such as vehicle cost, program delay cost (i.e., standing army), payload loss cost, redesign and requalification cost, loss of business cost, etc. It is estimated that the Challenger incident cost the U.S. 2 billion dollars. It might also be noted the cost of engines is small compared to this total figure. Therefore, cost models for catastrophic failures should concentrate on the other elements described above.

Then it is assumed that in this program in addition to catastrophic failures we are also interested in noncatastrophic failures and costs associated with them such as maintenance, program delay, safe engine shut down, and possible mission loss (but not vehicle). Just as unreliability has cost consequence, increased reliability would also have a cost consequence in terms of increased qualification testing costs as well as possible increase in payload to orbit cost because of increased engine weight. The ISB (Specific Impulse) has a negative sensitivity and engine weight has a positive sensitivity to payload cost.

The cost elements in the noncatastrophic failure modes should include:

1. Cost of removal of the component assembly and installation costs for new component assembly.
2. Cost of shipping of the component assembly to the plant if it is a factory repair.
3. Cost of teardown and replacement in terms of labor hours (a function of tear down level), labor cost, and new installation and material cost including cost of other mandatory replaceable items.
4. Green run test cost for the new component assembly.

Noncatastrophic failure in the engine level may still mean severe component failure in the subassembly level cascading through several parts in the same assembly or other adjacent assembly. Thus, the model should contain logic that links the components affected with their severity level so that a true cost consequence can be calculated. It must be realized that there is always some uncertainty involved in the cascading aspect of the failure scenario.

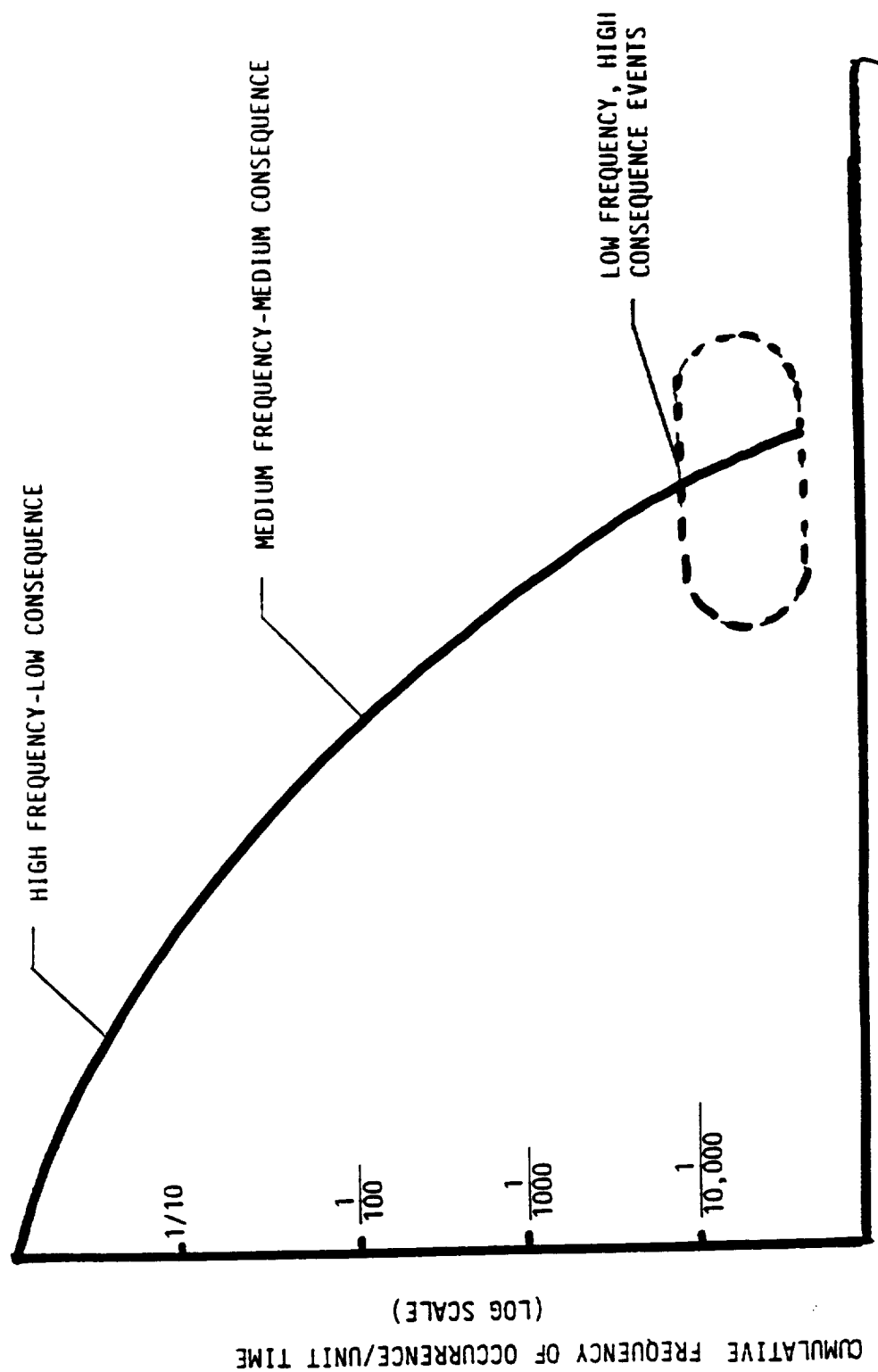
4 COST MODEL APPLICATION

A very large global model having the framework for other sensitivity studies in terms of total program cost is beyond the scope of the PSAM effort even though the basic reliability data from PSAM may be needed for use in the model. The total program cost management as a function of unreliability might require answers to very difficult questions such as:

1. What is optimum number of engines to build (cost per engine goes down as the number of units increase) to sustain a traffic pattern?
2. What should be the target reliability and optimum design life of the engine that will result in minimum cost?
3. Given the fact that refurbishment of an old engine involves high engineering time as opposed to a new engine, what is the optimum mix of old and new engines?

4. Given differing lives for a given reliability between many components and fixed budget, which component and in what quantity to build that will best sustain the ground test and flight traffic pattern?

Thus, it is advisable that PSAM concentrate on a small subset of cost models that contribute to structural failure risk management. In this regard it is expected that as part of PSAM application studies a few candidate problems from the SSME program will be studied in consultation with SSME Rocketdyne management and PSAM contract team members. One candidate problem any deal with ground test hardware and the other flight hardware. In ground test hardware, more risks are taken when compared to flight. In ground tests, the components are used to their design limits to satisfy the fleet leader requirements. Thus, a cost benefit analysis based on probability of failures is feasible. In flight hardware, the increased risk of failure when the rate power level is increased to 106% can be studied for identified components sensitive to the power level change.



DAMAGE S (LOG-SCALE)

FIGURE 1 RISK CURVES

APPENDIX E

A Fast Convolution Procedure for Structural Reliability Analysis

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A Fast Convolution Procedure for Structural Reliability Analysis

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Abstract

A probability analysis procedure is proposed that combines the convolution theorem with a fast probability integration concept. The convolution theorem provides an exact solution for a sum of random variables regardless of the shape of the distributions, and is suitable for engineering problems that involve well-behaved performance functions with non-normal, continuous random variables. The procedure consists of the following steps: (1) identify the most probable point of a limit state, (2) establish a quadratic surface around the most probable point, (3) transform the quadratic surface to a linear surface, and (4) apply a fast Fourier transform technique to provide a fast convolution solution. Several numerical examples are provided to demonstrate the procedure.

Introduction

A central problem in probabilistic engineering analysis is the computation of the cumulative distribution function (cdf) of one or more performance functions, $Z_j(X) = f(X_1, X_2, \dots, X_n)$ where X_i are the basic design (random) variables. Because performance functions are generally nonlinear and random variables are generally dependent, the convolution theorem is usually thought to be inapplicable to the problem. This report shows that the convolution theorem may be effectively used when applied in conjunction with a fast probability integration concept.

Fast probability analysis is desirable in complicated engineering design problems. In structural reliability analysis, efficient techniques have been developed based on a fast probability integration concept that involves three major steps: (1) define a limit state and find a good approximating point (e.g., a maximum likelihood point), (2) obtain an approximate function around the approximating point, and (3) perform probabilistic analysis based on the approximate function. The first two steps provide the basis for the effective use of the convolution theorem.

This report describes how to start from a "good" quadratic polynomial and perform a transformation to linearize the quadratic function. Subsequently, the convolution theorem is used to compute the probability. To speed up the convolution calculations, a discrete fast Fourier transform (FFT) technique is suggested. The report discusses short cuts in implementing the FFT technique.

Fast Probability Integration

The fast probability integration (FPI) concept originated from the field of structural reliability analysis. While the method requires that the performance functions be well-behaved, the concept is general and may be applied to a wide variety of engineering problems. In this section, we will describe the concept using the terms commonly used in structural reliability.

A limit state function is defined as:

$$g(\mathbf{X}) = Z(\mathbf{X}) - z = 0 \quad (1)$$

where z is a real value. The g -function separates the design space into "failure" and "safe" regions. From this definition, the cdf of Z at z , $F_Z(z)$, is the probability of $[g \leq 0]$. The cdf of Z can be established by varying z .

Given the joint probability density function, $f_{\mathbf{X}}(\mathbf{x})$, the probability of failure can be formulated as:

$$p_f = \int_{\Omega} \dots \int f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \quad (2)$$

where Ω is the failure region. This multiple integral is in general very difficult to evaluate. Alternatively, a Monte Carlo solution provides a convenient, but usually time-consuming approximation. For practical engineering applications, efficient approximate solutions are desirable and can be obtained by the recently developed structural reliability analysis methods [1, 2].

The first step in the reliability analysis involves the transformation of a generally dependent, random vector \mathbf{X} into an independent, standardized normal vector \mathbf{u} . For example, the Rosenblatt transformation has been suggested when the joint distribution is available [1, 2]. If only the marginal distributions and the covariances are known, a transformation can be made to generate a joint normal distribution that satisfies the given correlation structure [3].

By transforming $g(\mathbf{X})$ to $g(\mathbf{u})$, the most probable point (MPP) in the u -space, \mathbf{u}^* , is the point that defines the minimum distance, β , from the origin to the limit state surface, see Fig. 1. This point is most probable (in the u -space) because it has a maximum joint probability density on the limit-state surface.

The MPP (also called *design point*) is the key approximation point for the FPI analysis. The MPP can be obtained by formulating an optimization problem and by using proper optimization methods. If a performance function is defined implicitly and requires computer-intensive analysis, the procedure for identifying the MPP is critical. An effective procedure is proposed in Ref. [4] for such problems.

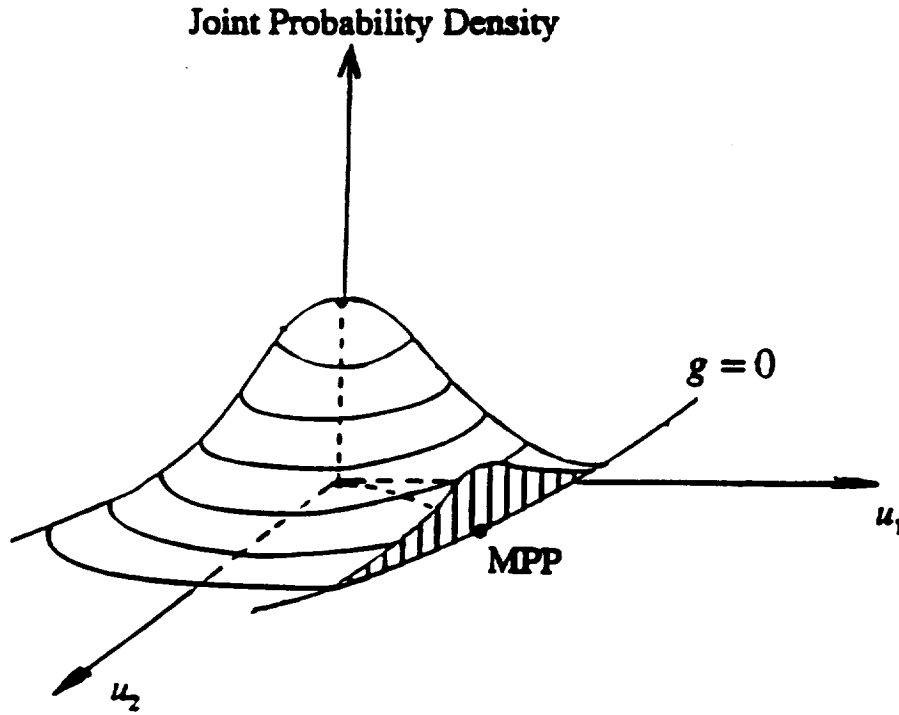


Figure 1. Illustration of a Most Probable Point (MPP)

Typically, the $g(\mathbf{u})$ or $g(\mathbf{X})$ function is approximated by a polynomial function that approximates the true function in the vicinity of the most probable point. Once the approximate function is obtained, the associated failure probability can be easily computed. If the $g(\mathbf{u})$ formulation is used, several analytical solutions are available for linear and quadratic $g(\mathbf{u})$ functions [5]. For example, the first-order reliability method (FORM) estimate is:

$$P(g \leq 0) \approx \Phi(-\beta) \quad (3)$$

and the asymptotic second-order reliability method (SORM) estimate, derived by Breitung [6], is:

$$P(g \leq 0) \approx \Phi(-\beta) \prod_{j=1}^{n-1} (1 + \beta \kappa_j)^{-1/2}, \quad \beta \rightarrow \infty \quad (4)$$

where $\Phi(\cdot)$ is the cdf of a standard normal distribution and κ_j , $j = 1, \dots, n-1$ are the main curvatures of the limit-state surface at \mathbf{u}^* .

Because the errors in an approximated g -function tend to be small around the more significant probability region, a fairly accurate solution may be achieved provided that the $g = 0$ surface is "smooth" and can be approximated by a quadratic surface in the u -space. This approach has been widely used for structural reliability analysis. However, when the approximate linear or quadratic g -functions involve non-normal random variables, no general closed form solutions are available. Although approximation methods have been developed (e.g., see Ref. 7), the accuracy tends to deteriorate when significantly non-normal variables are involved.

Fast Convolution Methodology

This section describes how to use the MPP to establish a linear or a quadratic polynomial and perform a transformation to linearize the quadratic function. Once the function is linearized, the convolution theorem and a FFT scheme can be applied to compute efficiently the probability.

Establish Linear or Quadratic Functions

The advanced mean value iteration algorithm leads to a most probable point. Based on the MPP, a first or second-order approximation function can be established in either the u - or the X -space using numerical differentiation methods or least-square curve-fitting methods. The minimum number of g -function calculations for establishing the first-order function is $(n + 1)$. A quadratic function without mixed or cross-product terms requires a minimum of $(2n + 1)$ function calculations. A full second-order function requires at least $(n + 1) \cdot (n + 2)/2$ function calculations.

Linearize Quadratic Functions

Based on the FPI concept, a method for linearizing quadratic Z -functions is described in Ref. [7]. A summary is described herein.

The first step is to approximate the Z -function by a second-order polynomial using Taylor's series expansion at the MPP, x^* ,

$$Z(X) = \sum_{i=1}^n a_i (X_i - x_i^*) + b_i (X_i - x_i^*)^2 \quad (5)$$

where a_i and b_i are coefficients. After rearranging terms,

$$Z(X) = \sum_{i=1}^n -\frac{a_i^2}{4b_i} + \sum_{i=1}^n b_i \left[X_i - \left(x_i^* - \frac{a_i}{2b_i} \right) \right]^2 \quad (6)$$

which may be transformed to a linear form

$$Z(Y) = c_0 + \sum_{i=1}^n c_i Y_i \quad (7)$$

where

$$Y_i = X_i - \left(x_i^* - \frac{a_i}{2b_i} \right)^2 \quad (8)$$

Eq. (8) defines the transformation from X_i to Y_i , where Y_i is a function of X_i only. Thus, $Z(Y) = 0$ is linear; Y_i 's are independent if X_i 's are independent. Given the pdf of X_i , the pdf of Y_i can be easily computed.

Remarks

- A quadratic equation can be established by a least-square method. If proper data are used, this method could reduce the error due to the exclusion of the mixed terms.
- In some cases, a log-transformation, i.e., $X_i' = \ln X_i$, for some or all of the X_i s may be preferred.

Further discussion can be found in Ref. [7].

Eq. (5) does not involve mixed terms. However, in the u -space, the mixed terms effect can be included by the following procedure (Ref. 8). Let a full quadratic function be computed in the u -space at the design point \mathbf{u}^* as follows:

$$\begin{aligned} g(\mathbf{u}) = & g(\mathbf{u}^*) + \sum_{i=1}^n \frac{\partial g(u)}{\partial u_i} \Big|_{\mathbf{u}^*} (u_i - u_i^*) \\ & + \frac{1}{2} \left[\sum_{i=1}^n \frac{\partial^2 g(u)}{\partial u_i^2} \Big|_{\mathbf{u}^*} (u_i - u_i^*)^2 \right. \\ & \left. + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{\partial^2 g(u)}{\partial u_i \partial u_j} \Big|_{\mathbf{u}^*} [(u_i - u_i^*)(u_j - u_j^*)] \right] = 0 \end{aligned} \quad (9)$$

or, in matrix notation, after some arrangements:

$$g(\mathbf{u}) = 2g(\mathbf{u}^*) + 2\mathbf{g}_u^T(\mathbf{u} - \mathbf{u}^*) + (\mathbf{u} - \mathbf{u}^*)^T \mathbf{G}_u (\mathbf{u} - \mathbf{u}^*) = 0 \quad (10)$$

where \mathbf{g}_u is the first-order gradient vector, and \mathbf{G}_u is the second-order derivatives matrix. This equation can also be rewritten as follows:

$$g(\mathbf{u}) = \mathbf{u}^{*T} (\mathbf{G}_u \mathbf{u}^* - 2\mathbf{g}_u) + 2\mathbf{u}^T (\mathbf{g}_u - \mathbf{G}_u \mathbf{u}^*) + \mathbf{u}^T \mathbf{G}_u \mathbf{u} = 0 \quad (11)$$

To eliminate the mixed terms, apply a rotational transformation

$$\mathbf{u} = \mathbf{R}\mathbf{v} \quad (12)$$

where the vector \mathbf{v} represents new standardized normal variables, \mathbf{R} is the modal matrix for G_u , and $\mathbf{R} \cdot \mathbf{R}^T = \mathbf{I}$ (identity matrix). G_u can be represented as follows:

$$G_u = \mathbf{R}\mathbf{\Lambda}\mathbf{R}^T \quad (13)$$

where $\mathbf{\Lambda}$ is the diagonal matrix of the eigenvalues of G_u . By substituting Eq. 12 into Eq. 11, Eq. 11 becomes

$$g(\mathbf{v}) = \mathbf{v}^{*T}(\mathbf{\Lambda}\mathbf{v}^* - 2\mathbf{g}_v) + 2\mathbf{v}^T(\mathbf{g}_v - \mathbf{\Lambda}\mathbf{v}^*) + \mathbf{v}^T\mathbf{\Lambda}\mathbf{v} = 0 \quad (14)$$

Thus there are no mixed terms in the \mathbf{v} -space. If G_u is regular, the eigenvalues, $\lambda_i, i = 1, 2, \dots, n$ must not all equal to zero. In this case, Eq. 14 can be written as

$$g(\mathbf{v}) = \delta_0 + \sum_{i=1}^n \lambda_i (v_i - \delta_i)^2 = 0 \quad (15)$$

where $\delta_i, i = 0, 1, 2, \dots, n$ are constants.

If G_u is singular, then one or more eigenvalues, $\lambda_i, i = 1, 2, \dots, m$ are equal to zero. Eq. 14 becomes a cylindrical equation as follows:

$$g(\mathbf{v}) = \delta_0 + \sum_{i=1}^m \lambda_i (v_i - \delta_i)^2 + 2 \sum_{i=m+1}^n v'_i g_{v_i} = 0 \quad (16)$$

where \mathbf{v}' represents those components that have $\lambda_i = 0$, and \mathbf{g}_{v_i} is the first-order gradient vector.

Once Eq. 15 or Eq. 16 is obtained, they can be linearized using the procedure described in Eqs. 5 to 8. This procedure uses a full second-order function in the \mathbf{u} -space to produce an exact SORM solution. The Breitung's SORM solution (Eq. 4) is an approximate SORM solution because it is based on a parabolic approximation function which is obtained by a rotation of the \mathbf{u} -axis followed by an orthogonal transformation.

Fast Convolution

Assuming that the Z function has been linearized and properly scaled such that it can be expressed as:

$$Z(\mathbf{X}) = \sum_{i=1}^n X_i \quad (17)$$

The characteristic function of a random variable X is:

$$H_X(\omega) = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx \quad (18)$$

which is the Fourier transform of the pdf, $f_X(x)$. It is also known that the characteristic function of a sum of independent random variables equals the product of the characteristic function, i.e.,

$$H_Z(\omega) = \prod_{i=1}^n H_{X_i}(\omega) \quad (19)$$

Equation (19) is useful because it does not involve multiple (nested) integrations, and therefore, is particularly convenient for large n . The pdf of Z is

$$f_Z(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_Z(\omega) e^{-jz\omega} d\omega \quad (20)$$

from which the cdf can be computed.

Although one can use numerical integration based on Eqs. (19) and (20), computational time can be substantially reduced by applying a discrete FFT scheme. This allows quick design analysis. In particular, the efficiency facilitates probabilistic sensitivity analysis in which the significance of each random variable to the performance cdf is evaluated by changing the distribution parameters (e.g., mean and standard deviation). To implement the fast convolution method, the range and the number of points for FFT must be carefully considered.

FFT Range

In applying a FFT technique [e.g., see Ref. 9], each pdf can be considered as a time varying "signal." Because in FFT each signal is treated as periodic, the range (signal length) must be wide enough to avoid the wrap-around phenomena. Consider a Z -function: $Z = X_1 + X_2$. In Fig. 2, the two pdf's have ranges $R_1 = B$ and $R_2 = D$. If the FFT range, R_o , is less than $R_1 + R_2$, the resulting pdf for Z would be incorrect because of the periodic assumption in FFT. This means that the minimum range is

$$R_o = \sum_{i=1}^n R_i \quad (21)$$

where each R_i is a range that "sufficiently" defines the pdf. Because a large R_o means that more points are required to model the pdf shape sufficiently, R_i should be chosen as small as possible to maximize the computational efficiency.

Referring to Fig. 2, the range may be reduced if the two signals are shifted such that $A = 0$ and $C = 0$. This implies that the bounds must be defined first. One way of defining bounds is to truncate the distribution at both tails using a probability tolerance, ϵ_i . For each random variable X_i , the lower bound (x_{iL}) and upper bound (x_{iU}) are defined as

$$1.0 - F_{X_i}(x_{iU}) \leq \epsilon_i \quad (\text{Right tail})$$

$$F_{X_i}(x_{iL}) \leq \epsilon_i \quad (\text{Left tail}) \quad (22)$$

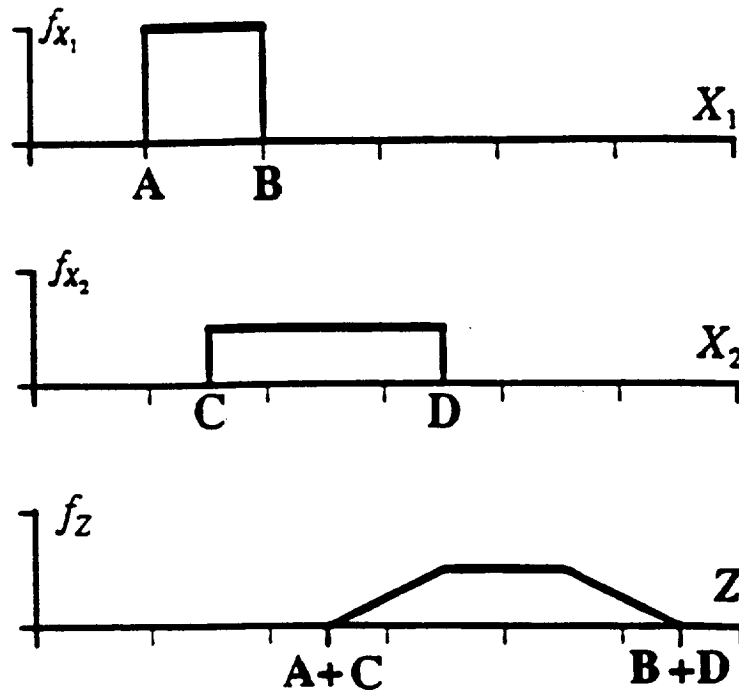


Figure 2. Selection of Range For FFT

Because there is no "signal" before x_{iL} , the following transformation can be performed to allow each signal to start at "time" zero.

$$Y_i = X_i - x_{iL} \quad (23)$$

The new pdf is truncated and ranges from 0 to $R_i = x_{iU} - x_{iL}$. Define a new performance function Z' as:

$$Z' = \sum Y_i = \sum X_i - \sum x_{iL} = Z - \sum x_{iL} \quad (24)$$

Using the Z' coordinate, the FFT range becomes:

$$R_o = \sum_{i=1}^n x_{iU} - x_{iL} \quad (25)$$

In defining the bounds, the tolerances ϵ_i should be selected such that the resulting error due to the truncations is sufficiently small. Probability error bounds can be computed by summing relevant tail probabilities. In evaluating the entire cdf involving a large n , the total range based on Eq. (25) may be too wide. To minimize the FFT range, the following strategy can be used.

Consider a probability of failure analysis in which p_f is small (say, $p_f < 0.1$). In such a case, the range can be substantially reduced. Let $g = Z' - z'$, where z' is a value at the left tail of Z' computed based on Eq. (24) and z . A conservative range is

$$R_o = n z' \quad (26)$$

This is because all Y_i are positive and any $Y_i > z'$ would not contribute to $[g < 0]$. The FFT range based on Eq. (26) is relatively narrow because usually $z' \ll R_i$.

The above strategy can also be applied when $F_Z(z)$ is large (near 1). In such a case, a new Z -function should be defined as $Z'' = -Z$ and $F_{Z''}(z) = 1 - F_Z(z)$ (a small probability) is calculated using the FFT technique.

If $R_i < z'$, the FFT range can be reduced, further, to:

$$R_o = \sum_{i=1}^n \min.(R_i, z') \quad (27)$$

FFT Points

The number of FFT points, N_p , should be large enough to describe the shape of the pdf's sufficiently. In general, the number of points used for any variable, within each range R_i , should be greater than a minimum number, i.e.,

$$N_i \geq N_{\min} \quad (28)$$

If any one of the variables violates the above requirement, the total number of points should be increased. However, the minimum number N_{\min} is related to the shape of the pdf and the machine accuracy and cannot be generalized. A more convenient approach is to, beginning with a small number (say, 128), repeatedly double the number of points until the solution converges to within a tolerance.

Illustration Example

To illustrate the above procedure, a simple strength-stress (R - S) problem is used. The Z -function is:

$$Z(X) = R - S$$

The random variables are independent with the following distributions:

$$R \sim \text{Normal} (\text{mean} = 25 \text{ ksi; std. dev.} = 4.)$$

$$S \sim \text{Normal} (\text{mean} = 10 \text{ ksi; std. dev.} = 3.)$$

The step-by-step procedure for obtaining the cdf of Z is illustrated using Fig. 3 and described as follows:

1. Obtain a linear performance function, Z , such that,

$$Z(X) = \sum_{i=1}^n X_i$$

where X_i are

$$X_1 = R \sim \text{Normal} (25., 4.)$$

$$X_2 = -S \sim \text{Normal} (-10, 3.)$$

2. Define the range for each X_i using $\epsilon_i = 10^{-4}$. The results are shown in Table 1.

Table 1

Random variable	Lower bound	Upper bound	Range
X_1	2.552	47.448	44.896
X_2	-26.836	6.836	33.672

3. Perform transformation of the random variables (Eq. 23):

$$Y_i = X_i - x_{iL}$$

4. Define the new function Z' :

$$Z' = Z - \sum_{i=1}^n x_{iL} = Z + 24.284$$

5. The FFT range for calculating the cdf of Z' is

$$R_o = \sum_{i=1}^n R_i = 78.568$$

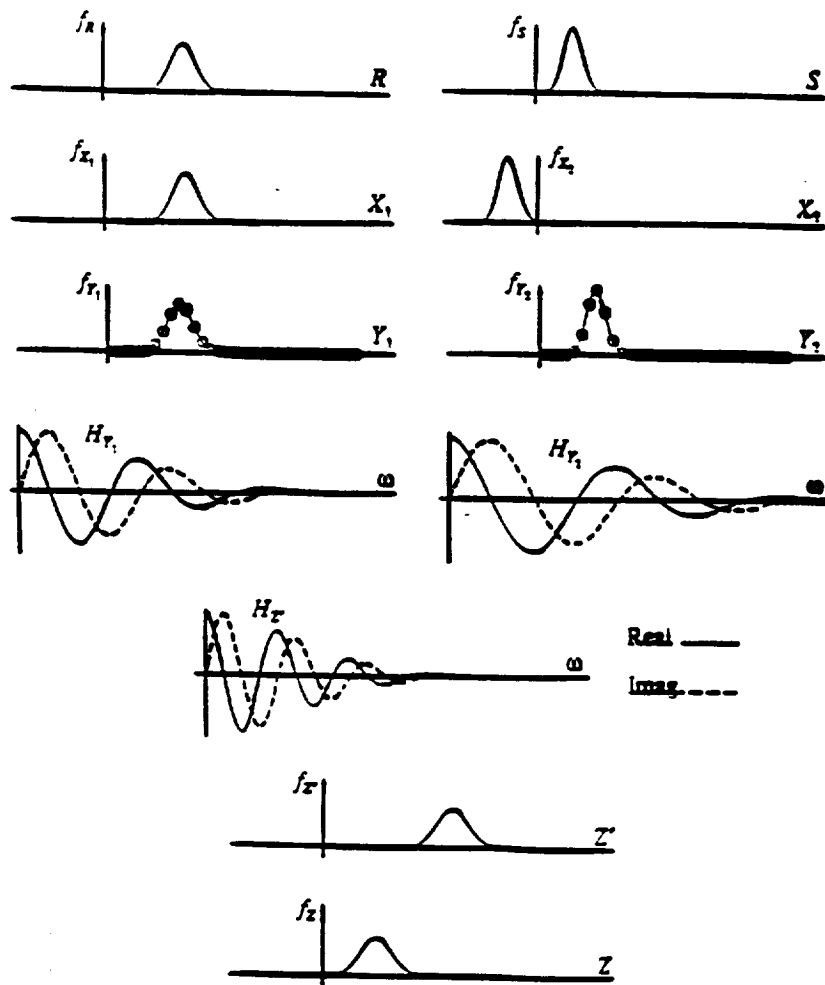


Figure 3. Illustration of the Fast Convolution Procedure

6. Select the number of points. As an initial try, use $N_p = 128$. This results in $N_1 = 74$ and $N_2 = 55$. (In Fig. 3, $N_p = 32$ is used.)
7. Perform discrete FFT for each Y_i
8. Multiply the results from step 7.
9. Perform an inverse FFT to obtain the pdf of Z' .
10. The pdf of Z is calculated based on

$$Z = Z' - 24.284.$$

Numerical Examples

A computer program was developed that implements the above analysis procedure. Seven sample problems used to test the capability of the proposed procedure are presented below. To simulate complicated structural reliability problems, we assume that the g -functions are not known explicitly and are "defined" in computer programs.

Once the MPP is identified, the second step is to calculate the probability of failure, p_f , by using several methods including the fast convolution method (FCM). A full, second-order g -function in the u -space and a partial quadratic g -function in the X -space are generated whenever applicable. Monte Carlo (MC) solutions based on the exact g -functions are generated for comparisons.

Example 1

The g -function is

$$g(X) = R - L - D$$

in which R , L and D are random variables having the following distributions (mean, standard deviation):

$R \sim \text{Weibull}(50, 5)$

$L \sim \text{Type I asymptotic extreme}(10, 2)$

$D \sim \text{Lognormal}(20.223, 3.033)$

The reliability analysis results using several methodologies are summarized in Table 2. Since the limit state is linear in the X -space, the fast convolution method in the X -space should produce an "exact" solution. The results indicate that all the methods produce good solutions except FORM.

Table 2. Reliability Results of Example 1

Method	FORM	SORM (Breitung)	FCM (<i>u</i> -space)	FCM (<i>X</i> -space)	Exact (MC)
Probability of failure	2.693e-3	3.502e-3	3.510e-3	3.571e-3	3.558e-3

Example 2

The *g*-function, taken from Ref. 1, is:

$$g(X) = YZ - M$$

in which the three independent random variables have the following distributions:

$Y \sim \text{Lognormal (mean} = 40 \text{ ksi; Cov} = 0.125)$

$Z \sim \text{Lognormal (mean} = 50 \text{ in}^3; \text{Cov} = 0.05)$

$M \sim \text{Type I EVD (mean} = 1000 \text{ in-kips; Cov} = 0.2)$

The reliability results are summarized in Table 3. The results again indicate that all the methods are good except FORM.

Table 3. Reliability Results of Example 2

Method	FORM	SORM (Breitung)	FCM (<i>u</i> -space)	FCM (<i>X</i> -space)	Exact (MC)
Probability of failure	3.051e-3	3.095e-3	3.146e-3	3.136e-3	3.129e-3

Example 3

The *g*-function is:

$$g(X) = X_1^2 + X_2 - 8.7577$$

in which the two independent random variables have the following distributions:

$X_1 \sim \text{Normal (mean} = 10; \text{Cov} = 0.30)$

$X_2 \sim \text{Normal (mean} = 10; \text{Cov} = 0.30)$

The reliability analysis results are summarized in Table 4. Since the *g*-function is quadratic, the FCM method should produce an "exact" solution. The results indicate that SORM (Breitung) is less accurate because of the parabolic approximation.

Table 4. Reliability Results of Example 3

Method	FORM	SORM (Breitung)	FCM (u -space)	FCM (X -space)	Exact (MC)
Probability of failure	1.11e-3	8.42e-4	6.89e-4	6.89e-4	7.04e-4

Example 4

This example represents one of the problems for which the mixed terms are important. The g -function is:

$$g(\mathbf{X}) = X_1^2 + X_2^2 - 2X_1X_2 + X_1 + X_2$$

in which the two independent random variables have the following distributions:

$$X_1 \sim \text{Normal (mean} = 10; \text{Cov} = 0.50)$$

$$X_2 \sim \text{Normal (mean} = 10; \text{Cov} = 0.50)$$

The reliability analysis results are summarized in Table 5. The results indicate that FCM (X -space) produces substantial error. The reason is because the mixed terms are neglected. Fig. 4 shows the substantial error in the incomplete (without mixed terms) quadratic function in the u -space. The FCM (u -space) produces near exact solution because the g -function is quadratic in the u -space. The FORM solution is the worst, while the SORM (Breitung) solution is good.

Table 5. Reliability Results of Example 4

Method	FORM	SORM (Breitung)	FCM (u -space)	FCM (X -space)	Exact (MC)
Probability of failure	2.339e-3	3.653e-4	3.248e-4	1.201e-4	3.304e-4

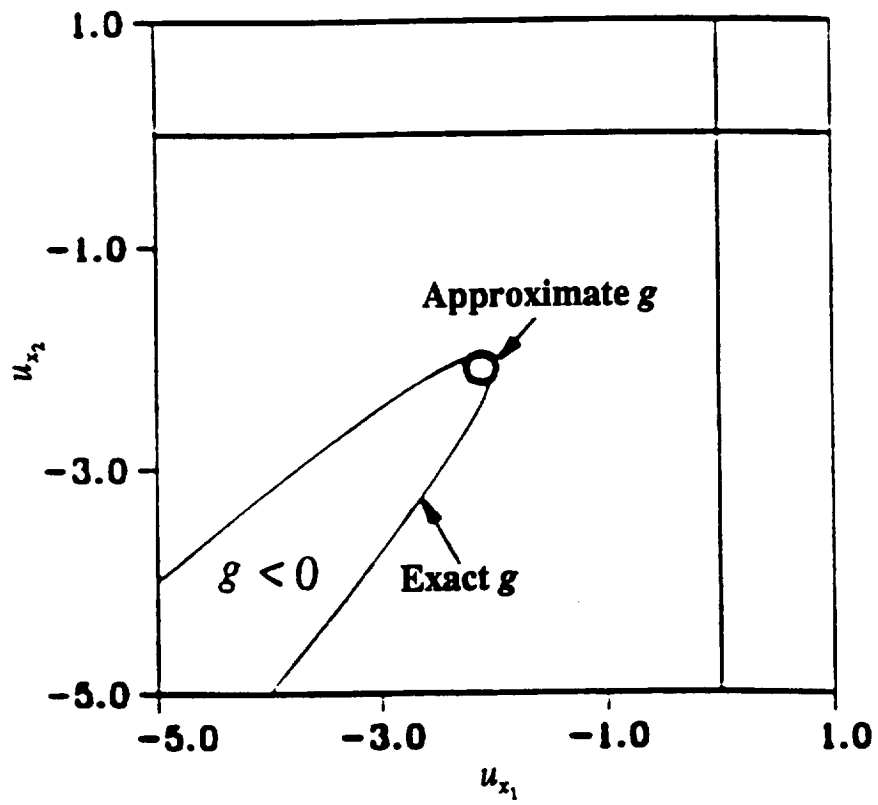


Fig. 4. Illustration of Exact and Approximate Limit States for Example 4

Example 5

This example represents one of the problems for which the normal transformation distorts the original limit state drastically such that the quadratic limit state is not a good approximation.

The g -function is:

$$g(X) = R - S$$

where

$$R \sim \text{Lognormal} (\text{mean} = 20; \text{Cov} = 0.25)$$

and S has a bi-modal pdf defined as:

$$f_S(s) = (0.99) \cdot \phi\left(\frac{s - \mu_1}{\sigma_1}\right) + (0.01) \cdot \phi\left(\frac{s - \mu_2}{\sigma_2}\right)$$

where

$$(\mu_1, \sigma_1, \mu_2, \sigma_2) = (10, 2, 40, 2)$$

The reliability analysis results are summarized in Table 6. Fig. 5 shows the exact limit state and the full second-order approximation in the u -space. The FCM (X -space) method produces near exact solution because the g -function is linear. The results suggest that the g -function in the u -space cannot be adequately approximated by a quadratic function because of the nonlinear normal transformation.

Table 6. Reliability Results of Example 5

Method	FORM	SORM (Breitung)	FCM (u -space)	FCM (X -space)	Exact (MC)
Probability of failure	1.567e-2	1.517e-2	1.507e-2	2.347e-2	2.307e-2

Example 6

The limit-state function, taken from Ref. 10, is:

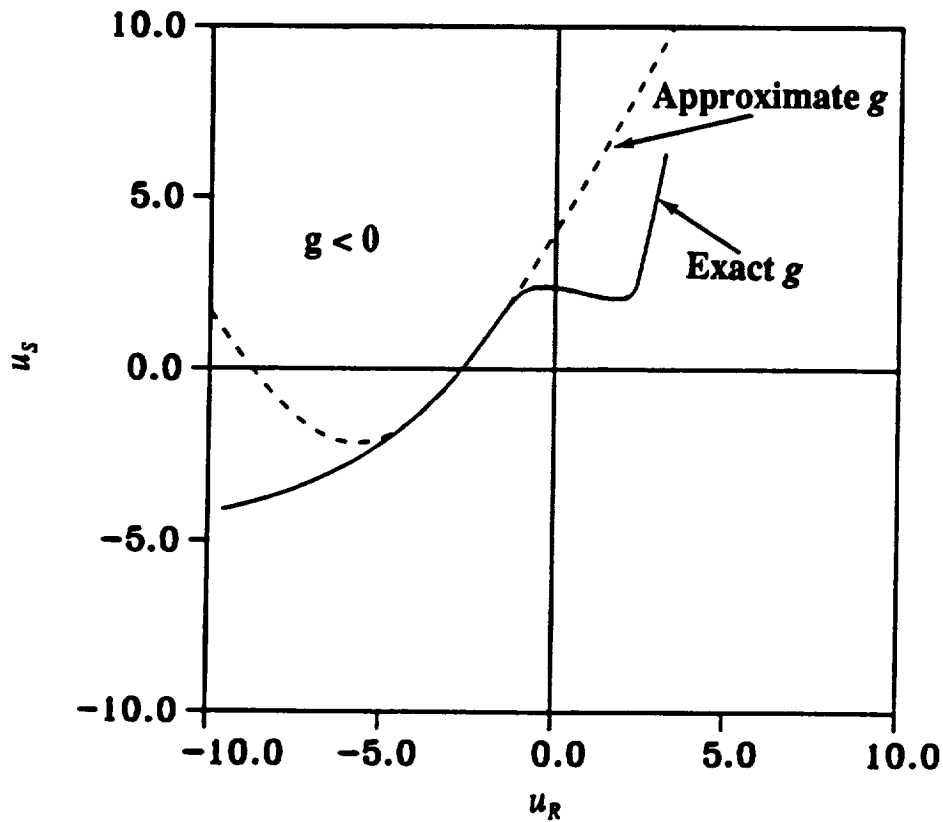


Fig. 5. Illustration of Exact and Approximate Limit States for Example 5

$$g(\mathbf{X}) = C - X_1 - X_2 = 0$$

in which C is a constant and X_1 and X_2 are exponentially distributed with cdf's of

$$F_{X_i}(x_i) = 1 - e^{-x_i} \quad i = 1, 2$$

The coefficient of variations are 100% for both X_i . The p_f results are shown in Table 7 in which FORM is the result of the first-order reliability method [1, 2], $p_f = \Phi(-\beta)$, where $\Phi(\cdot)$ is the standard normal cdf and β is the minimum distance described earlier.

Table 7A Reliability Results of Example 6 ($C = 11$)

Method	FORM	SORM (Breitung)	FCM (<i>u</i> -space)	FCM (<i>X</i> -space)	Exact (MC)
Probability of failure	9.191E-5	2.1898E-4	2.233E-4	2.036E-4	2.018E-4

Table 7B Reliability Results of Example 6 (*C* = 13)

Method	FORM	SORM (Breitung)	FCM (<i>u</i> -space)	FCM (<i>X</i> -space)	Exact (MC)
Probability of failure	1.359E-5	3.502E-5	2.701E-5	3.184E-5	3.234E-5

The FORM result is poor because the random variables are significantly non-normal. The convolution approach in *X*-space, in theory, should produce an exact solution because the *g*-function is linear.

Example 7

This problem represents one of the conditions for which the FORM approximation is totally unsatisfactory. The limit state, taken from Ref. 10, is a hyper-sphere:

$$R^2 = u_1^2 + u_2^2 + \dots + u_n^2$$

where all u_i 's are standard normal variates. The minimum distance is the radius of the hyper-sphere, *R*. Using the transformation:

$$Y_i = u_i^2$$

the limit state can be linearized as:

$$g(\mathbf{Y}) = R^2 - Y_1 - Y_2 - \dots - Y_n = 0$$

where each Y_i is a chi-square distributed variable with one degree of freedom. The exact solution is:

$$p_f = 1 - \chi_n^2(R^2)$$

where $\chi_n^2(\cdot)$ is the cdf of the chi-square distribution with *n* degrees of freedom. The p_f results for *n* = 5 are shown in Table 8.

Table 8A Reliability Results of Example 7 (*R* = 4)

Method	FORM	SORM (Breitung)	FCM (u -space)	FCM (X -space)	Exact (MC)
Probability of failure	5.965E-4	2.909E-2	1.807E-2	6.574E-3	6.8E-3

Table 8B Reliability Results of Example 7 ($R = 5$)

Method	FORM	SORM (Breitung)	FCM (u -space)	FCM (X -space)	Exact (MC)
Probability of failure	6.234E-6	9.758E-4	5.256E-4	1.395E-4	1.4E-4

Only the fast convolution (X -space) approach produces good solutions.

Summary and Discussions

As demonstrated, the proposed procedure is capable of producing "exact" solutions for highly non-normal distributions or non-standard distributions (e.g., bi-modal distribution). The convolution procedure in combination with the fast probability integration concept provides a useful tool for probability analysis. The major source of errors in the proposed procedure is in the approximate performance functions. The fast convolution method is capable of producing "exact" solutions for quadratic g -functions. However, it should be emphasized that the convolution solution is only as good as the approximate g -functions in either the u - or the X -space.

In general, normal transformation (i.e., from X to u) tends to introduce nonlinearity into the limit state. Therefore, when highly non-normal (e.g., bi-modal distribution) random variables are involved, it is desirable to use $g(X)$ approximation. However, since the second-order mixed-terms cannot be included for fast convolution analysis, significant errors due to the neglected mixed-terms may be possible for some highly nonlinear $g(X)$ functions. When the random variables are not highly non-normal, full quadratic $g(u)$ approximate functions are recommended. Additional research is desired to develop a method to include the mixed-terms effect (in the X -space) in the fast convolution analysis framework and to develop a method for selecting $g(X)$ or $g(u)$ approximations.

Based on numerous test examples, we found that the numerical errors are typically less than 1% using a VAX 8700 machine which has 16 decimal digits precision. To investigate the accuracies

that can be achieved at the extreme tails of the cdf, a CYBER 175 machine which has 32 decimal digits precision was used. It was found that the range of accurate cdf values extends from $u = \pm 5$ to $u = \pm 7$ (u is a standard normal variate). The required CPU time is in the order of 10 seconds.

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APPENDIX F

Probabilistic Boundary Element Method (PBEM) User's Manual

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The sensitivity calculations required for probabilistic analysis are performed using a complete reanalysis procedure. Many methods exist that are relatively more efficient than the present approach; however, the reanalysis procedure used in the present version of PBEM is relatively simple, easy to implement, and easily adaptable with the new versions of BEST3D.

The present version of PBEM is capable of performing sensitivity analysis for all types of problems that BEST3D can handle including materially nonlinear problems. In the appendix, detailed documentation on how to use PBEM in performing a perturbation analysis is presented.

A typical input data set for PBEM has the following features.

1. Deterministic data set--same as the BEST3D data set.
2. The deterministic data set should be terminated by *PROB option, to indicate the starting of probabilistic data set.
3. Definition of the random variables.
4. Definition of the perturbations.

The data set has similar options as in PFEM.

***PROB**

This card should be used before the probability data deck to indicate the stating of probability data.

***DEFINE**

This option is used to define the random variable (in terms of the number assigned to a random variable), and it should be followed by the statements giving the mean and standard deviation, and the description of the random variables.

The input format for defining a perturbation variable is as follows

***DEFINE jpvar**

dmean dstdev

CNTL

.....

.....

..data block..

Where

jpvar	is the perturbation variable number.
dmean	is the mean value of the random variable.
dstdev	is the standard deviation of the random variable.
CNTL	is the actual case control card of BEST3D to recognize the perturbation variable type.
Data block	is a block of data specifying actual perturbations, and depends on the type of perturbation. The following section gives the details about these options.

The first set of definitions required for defining the perturbations is the case control card of BEST3D. They could be any one of the following for a perturbation number.

BCSE for boundary conditions perturbation.

MATE	for material data input perturbation.
BODY	for body force perturbation.
GMR	for geometric perturbations.

Two more parameters following the case control card should be given to specify the ID number of the case control card (body force, boundary conditions, material property, and GMR) and type of boundary conditions or modules or body force parameters (as in centrifugal acceleration etc.)

Boundary Condition Perturbations:

ID idno
type num
T t1 v1 v2 .. vn
T t2 v1 v2 .. vn

idno	is the actual boundary condition type, DISP, TRAC, FLUX, or TEMP.
num	is the component number of the DISP or TRAC, and blank for FLUX and TEMP.
t1,t2	time steps at which the boundary conditions are specified.
v1,v2..	are the rate of change of boundary condition for specified node locations (equivalent to the boundary conditions in BEST3D)
T	Signifies (as in BEST3D) the actual starting of the specified boundary condition.

Identification of the B.C. type:

DISP	displacement specified.
TRAC	traction specified.
FLUX	flux boundary condition specified.
TEMP	Temperature boundary condition specified.

Example:

*DEFINE 1	..1
100.0 10.0	..2
BCSE	..3
ID ENDload	..4
DIS 3	..5
T 1 1.0 0.5 0.1	..6

Variations in the Material Property Inputs:

The same variable names as in BEST3D are employed for this purpose.

EMOD	Young's Modulus.
POIS	Poisson's ratio.
ALPH	Coefficient of thermal expansion, if thermal body force exists.
DENS	Density: if inertia and centrifugal body forces are specified in the deterministic

input.
 TECH for perturbations in anisotropic material
 property.
 ORIE Material Orientation for anisotropic problem.

Example:

```
*DEFINE 3          ..1
3.e+07,1.e+5       ..2
MATE                ..3
ID MAT1             ..4
EMOD 1.0 0.8 0.1   ..5
```

Where the Young's modulus is specified at three different temperatures. Similarly, the anisotropic constants and material orientation properties should be specified. The statement number ..5 is to be changed depending upon the type of variable (Ref. BEST3D users manual).

Variation of Body Force Parameters:

In BEST3D, the following notations are used for identifying the body force type, which is continued in PBEM.

CENT	Centrifugal force.
INER	Inertia force.
THER	Thermal body forces.

Body Force Perturbation Types:

DIRE	Direction of axis of rotation or acceleration.
SPEE	Speed for centrifugal body force.
ACCE	Acceleration due to gravity.
GMR	Identification of GMR for thermal input. This card should be followed by the perturbation to nodal temperature and used only in conjunction with THERM bodyforce.

Example :

```
*DEFINE 1
100. 10.
BODY
ID CENT
DIRE 0.0 0.0 1.0
```

The last statement above gives the rate of change of direction cosines as the perturbation type. Similarly for DIRE, SPEE, and ACCE, the rate of change is prescribed in a format similar to the deterministic data set.

The format of the above input statements should be changed as follows for the thermal body force.

```
ID GMR1
TEMP
n1 t1 t2 t3 ..
n2 .....
```

Where GMR1 is the region for which temperature is specified. n1, n2 are the node numbers, and t1, t2, and t3 are the temperatures at three different time intervals.

Geometric Variations:

In any perturbation, only nodal coordinates of a GMR (A BEM zone used in BEST3D) are perturbed. Hence, it is sufficient to provide the rate of change of nodal coordinates values in a format similar to the deterministic data set.

Example :

```
*DEFINE 2
100. 10.
GMR
ID GMR1
POINTS
1 0.01 0.01 0.0
2 0.0 1.0 0.0
.....
.....
.....
n x11 x21 x31
```

where

n	node number
x11, x21 and x31	rate of change of nodal coordinates for design change.

***PERTURBATION**

This is similar to PFEM.

```
*PERT jpert
jpvar1 shift1
```

This option is used to define a perturbed problem, constructed by prescribing small changes to one or more perturbation variables.

jpert	corresponds to the perturbation number.
jpvar1	is the random variable number.
shift1	is the number of standard deviation by which the jpvar1 is changed.

Example:

```
*PERT 3
3 0.01 1
```

The above statements indicate that the third perturbation involves a change of 0.01 standard deviation in design variable number 3.

***END**

This is a card required for indicating the end of perturbation information data deck for PBEM.

Performing MVFO and AMVFO Using PBEM: (*MOV)

The present version of PBEM requires the manual development of FPI input deck for a particular performance variable, unlike the PFEM. This, however, is not the case with advanced mean value method. Once the mean value analysis is performed the solution for AMVFO can be found automatically by executing the MOV.EXE file. MOV.EXE file is written exclusively for reading the information from FPI.MOV file and moving the design points accordingly in the BEST3D input deck, and solving for the new value of the performance function for a particular probability level.

APPENDIX G

Expert System Documentation

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Analysis Type Help Screen

This option defines the analysis solution points and is specified to FPI with the *ANALTYPE keyword in the parameter data. Several ways of selecting the solution points are available. Note, the minimum probability level for which FPI will obtain a solution is -5 standard deviations; the maximum is +5 standard deviations.

Entire CDF (*ANALTYPE = 0)

FPI will perform a search procedure in an attempt to locate points spanning approximately -5 to +5 standard deviations. 9, 10 or 11 points will be chosen and results will be chosen for these points. The points will be approximately equally spaced along the response (or z) axis.

This search procedure can be time consuming and can sometimes cause numerical errors. If difficulties arise with this method the user should choose the Zlevel procedure (*ANALTYPE = 1).

This method can be used with Monte Carlo (*METHOD = 6), but not with importance sampling (*METHOD = 5,7).

Plevels (*ANALTYPE = 2)

FPI will compute the solution at user-specified probability levels. The probability value is entered in linear (0 to 1.0) format. When using this format, FPI first performs a search in order to span the specified range of probability. FPI then uses curve fitting to estimate the response corresponding to the specified probability. FPI then runs a Zlevel procedure at the predicted response value. If the predicted probability in terms of U is within .1 % of the specified value, FPI proceeds to the next P level. If the predicted probability is not within .1%, FPI iterates until convergence.

This procedure is very similar to computing the entire CDF; therefore, this search procedure can also be time consuming and can sometimes cause numerical errors. If difficulties arise with this method the user should choose the Zlevel procedure (*ANALTYPE = 1).

This selection method cannot be used with Monte Carlo.

Zlevels (*ANALTYPE = 1)

FPI will compute the solution at user-specified response levels (Z values). In contrast to the other two analysis selection methods, FPI will still estimate the probability level even if it is outside -5 to +5 standard deviations up to -10 to +10 standard deviations.

This is the failsafe procedure for operating FPI and should be used when others fail.

This method can be used with Monte Carlo with importance sampling (*METHOD = 5,7) but not conventional Monte Carlo (*METHOD = 6).

Confidence Interval Help Screen

Confidence intervals can be obtained on the CDF. These confidence intervals are with respect to uncertainties in the underlying parameters of the random variable statistics. Thus, if there is some uncertainty in the mean and standard deviation of a random variable, the uncertainty can be entered in the form of a coefficient of variation of the mean and standard deviation. 90% and 95% confidence limits will be obtained. The number of samples points to use for determining the confidence intervals is recommended to be 10,000.

Entire CDF Help Screen

The only option necessary is the FPI parameter data *ANALTYPE = 0. No model data is necessary. If problems arise using this selection method, try Zlevels.

G Function Help Screen

The G function option defines the equation which divides the probability region into failed and safe regions. There are several options for defining the G function in NESSUS which can be classified into two types, explicit g functions and implicit g functions.

Explicit performance functions, i.e., closed form.

Use *GFUNCTION = 0,10 or 6. The g function must be defined in a user-defined routines USER or RESPON.

To use subroutine USER, option 0 or 10, the closed form equation must be written in a special format, $x(1) = f(x(2), \dots, x(n))$. If this is not possible, then option 6, subroutine RESPON, must be used. In our experience, subroutine USER is the more numerically robust and should be used if possible. This option is limited to zlevels (*ANALTYPE = 1) analysis only.

Subroutine RESPON, option 6, is more general and allows the user more freedom in programming the g function. However, the user should be aware that he must select distribution parameters for the random variables that make sense or a numeric error may result. For example, the distribution for Poisson's ratio should not allow NU to be greater than 1/2 because meaningless results or program crashes may result. Program crashes such as square root of a negative number, etc., can often result when using this option, especially when using

analysis type 0 or 2. If this happens the user should first try a Zlevel procedure and also examine the input data to see if it is realistic. This option can be used with analysis selection types 0,1, or 2.

Implicit - *GFUNCTION = 1,2

This option is useful when coupling FPI with other complicated analysis engines which define the g function such as the finite element method.

The g function is defined by a set of input data points. The data points are input with the *DATASETS option. FPI performs a regression analysis to compute a linear, option 1, or quadratic, option 2, g function. The current version of the code is limited to 100 datasets. Note, for many datasets regression analysis can be performed outside of NESSUS and the resulting equation programmed as an explicit g function. Analysis of implicit g functions is most accurate when performed within the Advanced Mean Value algorithm.

Random Variable Help Screen

The random variable definitions in NESSUS require a mean, standard deviation and a distribution type. Under NESSUS 4.2, the data is input in two locations, the *DEFINE card in the FEM input data and the *DEFRANV keyword in the FPI input data. The random variable definitions between the two keywords often coincide, but not always. The FEM random variables are limited to FEM quantities such as properties, pressures, etc. Therefore, the *DEFINE cards may be intermediate or psuedo random variables. The *DEFRANV random variables are always the independent random variables.

To assist the user in defining the random variable parameters, a list of default distribution parameters has been compiled. These can serve as ballpark estimates if problem specific data is not available.

Plevels help screen

The FPI parameter data *ANALTYPE = 2 must be input. The number of probability levels and their values must be input in the model data section using the *PLEVELS keyword. If problems arise using this selection method, try Zlevels.

Solution Method Help Screen

There are two basic solution methods in NESSUS, fast probability integration and Monte Carlo. The solution method is selected with the *METHOD parameter keyword.

The fast probability integration methods have two options, the first order reliability method, option 0, and the advanced first order reliability method, option 1. This is more accurate but more time consuming. In addition, for some problems numerical errors can arise with this method which don't show up for the first order reliability method. The advanced method should always be tried first.

Monte Carlo methods are an alternative solution method. Conventional Monte Carlo and importance sampling (Harbitz method) are included in NESSUS. Importance sampling is more efficient for obtaining the probability at a single response point, but is not designed to compute the entire CDF. Importance sampling can only be used at specified response levels. If the entire CDF is desired, conventional Monte Carlo should be used. One solution strategy is to compute the CDF between -3 to +3 standard deviations with conventional Monte Carlo and compute selected points in the tails with importance sampling. Note from the FPI theoretical manual that the efficiency of importance sampling is very high for few random variables, but degrades rapidly as the number of random variables increases.

Zlevels help screen

The FPI parameter data *ANALTYPE = 1 must be input. The number of response levels and their values must be input in the model data section using the *ZLEVELS keyword. This is the most failsafe method.

APPENDIX H

NESSUS/SHELL Knowledge Base User Interface

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NESSUS/SHELL KNOWLEDGE BASE USER INTERFACE

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There are three parts of the data decks connected with probabilistic analysis that are to be generated by NESSUS/SHELL. One is the deterministic data deck *FEM part two in the input data deck). The second part is the data deck *PFEM (part one in the input data deck) and the third part is the data deck *FPI (part three in the input data deck). The task is to prepare as much of the probabilistic data deck as possible, using the data deck for the deterministic problem as part fo the necessary knowledge base. The probabilistic data deck is to be inferred from the deterministic data deck, using the logic of NESSUS and past experience (expert knowledge) in creating probabilistic data decks.

1 Part one: the data in *FEM deck

This part of the data is in the *FEM data deck. Most of new data structures are connected with perturbation of the random variables. These are:

- a. data keyword *PERTURB in parameter data section
- b. data keyword *PERTURB in model data section
- c. data keyword *DEFINE in model data section
- d. data keyword *MOVE in model data section
- e. data keyword *RECORD in model data section

1.1 Keyword *PERTURB in parameter data section

This keyword will define how many random variables will be taken into consideration and what is variable. The form of this keyword is as follows

```
*PERTURB npvar (number of perturbation variables) npert (number of perturbations)
TYPE1 (TYPE of random variable)
TYPE2
.....
TYPEn
```

where the number of perturbation variables and the number of perturbations may be input by users (usually the number of perturbation variables is not equal to the number of TYPE). The knowledge base system can give a suggestion on the minimum number (the number of random variables) of perturbations. The number of perturbations depends on the number of random variables and the choice of FPI method (for first-order mean value method it should be at least equal to the number of variables, for second-order method it should be at least equal to one plus two times the number of variables). Selection of a large number of perturbations will cost more computer time, but the result could be more accurate. Usually the first order mean value method will be used, so in this system only the first order mean value will be taken into consideration.

The SHELL will define all the TYPE of random variables automatically after all random variables are defined by users. The TYPE quantities which can be selected are controlled by the type of the problem prescribed in the deterministic data deck. For instance if the problem is a static analysis, the choice of ACCELERATION, VELOCITY and HARMONIC are not appropriate and

will not appear in the choice menu. For two or three dimensional problems the choice of BEAMSECTION, which is only valid for element 98, will not appear in the choice menu, etc. Thus, SHELL can use the part of the knowledge base represented by the deterministic data deck.

In the *PERTURB keyword, there are seventeen TYPE of random variables that may appear in the choice menu:

ACCELERATION	Initial acceleration (dynamic problem).
BEAMSECTION	Beam section properties (just for beam element).
COORDINATES	Mesh geometry (any problem).
DAMPING	Damping constants (dynamic problem with damping option).
DISPLACEMENT	Initial displacement (any problem except eigenvalue problem).
DISTRIBUTEDLOAD	Element distributed loads (any problem except eigenvalue problem).
FORCES	Nodal forces (any problem except eigenvalue problem).
HARMONIC	Harmonic excitation parameters(dynamic problem with harmonic option)
ORIENTATION	Material orientation (any problem).
PRESSURE	Nodal pressures (any problem except eigenvalue problem).
PROPERTIES	Material properties (any problem).
PSD	Power spectrum excitation (dynamic problem with PSD option).
SPRINGS	Base spring stiffness (any problem).
TEMPERATURE	Nodal temperatures (any problem).
UPERT	User defined loading perturbation (any problem).
VELOCITY	Initial velocity (dynamic problem).
YIELDFUNCTION	Yield function curves (plastic problem).

1.2 Keyword *PERTURB in model data section

The data in this keyword connected with the data in this keyword *DEFINE will determine how large the perturbation will be taken. The form of this keyword is

```
*PERTURB jpert (the perturbation number)
jpvar1 (the random variable number) shift1 (the shift number)
jpvar2 shift2
.....
jpvarn shiftn
```

Usually this keyword can be prepared from the SHELL knowledge base, because the shift number for the first n perturbations can be set to default value 0.1 for most cases; if the second-order FPI method is chosen or npert in *PERTURB in parameter section is set larger than the number of random variables, the shift number in the following perturbations should be set to default value -0.1. In every perturbation process just one random variable is taken into consideration, so the jpert and jpvarn will just take the serial number of random variable or jpert will be set to a serial number

larger than random variables for the choice of larger npert in *PERTURB or the second-order FPI method.

1.3 keyword *DEFINE in model data section

This data keyword connected with the knowledge base data in keyword *PERTURB in the model data section will define how large the perturbation of the random variables will be taken.

Every part of these data has the similar form

*DEFINE jpvvar (a serial random variable number)

dmean (mean value) dstdev (standard deviation)

TYPE (see the Table above) jparam (optional for some random variables)

DATA BLOCK (a data block)

Obviously, all the mean values of the random variables and jparam should be the same as the value given for the deterministic problem. Thus, the mean values will come directly from the deterministic data section. The standard deviation for every random variable should be input by the users or specified by a default value dictated by expert opinion.

The following table concerns the input of the data in data block

TYPE	dmean	dstdev	jparam	Data Block
ACCELERATION	k	i	k	p
BEAMSECTION	k	i	k	p
COORDINATES	k	i	k	p
DAMPING	k	i	k	p
DISPLACEMENT	k	i	k	p
DISTRIBUTEDLOAD	k	i	k	p
FORCES	k	i	k	p
HARMONIC	k	i	k	p
ORIENTATION	k	i	k	p
PRESSURE	k	i	k	p
PROPERTIES	k	i	k	p
PSD	k	i	k	p
SPRINGS	k	i	k	p
TEMPERATURE	k	i	k	p
UPERT	k	i	k	p
VELOCITY	k	i	k	p
YIELDFUNCTION	k	i	k	p

k: known from the deterministic knowledge base

i: user override of default available (by input the value of COV)

p: partly input by user, using expert advice.

The most difficult thing is the data block which defines the total change of the random variable for perturbation, i.e., if the perturbation shift number in *PERTURB keyword is given and the corresponding number in the data block is k, then the total change for this corresponding variable will be $(k \cdot \text{shift} \cdot \text{dstdev})$. In most cases the number k will be set to one. But, for the perturbation of the coordinates, users will be required to input these data.

The format of the data blocks should be predetermined by the format in the deterministic data deck, users don't need to be asked to input every component data in the data block (most of them will be set to zero automatically). However the node where this random variable will be perturbed must be input by the user. In general two pieces of information should be input by the user in a block data:

- a. the nodes where the random variable will be perturbed.
- b. which component in this group of data is random variable.

For instance, if the Poisson's ratio (which is the third component in property data group *PROPERTY) from node 1 to node 34 should be perturbed, the following data block will be generated.

1 34 0. 0. 1. 0. 0. 0.

and so on (usually the perturbation value is default to all nodes).

Usually the data block under this keyword is the largest data block for probabilistic analysis.

1.4 keyword *MOVE in model section

This keyword is to define a new unperturbed problem, i.e., a new deterministic problem for which a shift of the perturbed variables will be computed; additional perturbations centered about the new unperturbed (with the shifted value of random variable) state may be computed.

The format of this keyword is

*MOVE

jpvar1 (the number of random variable) shift1 (the shift number)

jpvar2 shift2

.....

jpvarn shiftn

The shifted random variable will equal to $\text{shift} \cdot \text{dstdev}$. Usually, the user will not use this keyword. The value of shift could be set to one or be input by the users. If *PFEM module is included in the input data file, this keyword will be automatically inserted by SHELL.

1.5 keyword *RECORD in model section

This keyword is used to select which results should be stored and/or updated in the NESSUS perturbation database. It seems advisable to use the default values (all of the results obtained in the course of a given run are stored and/or updated in the database).

2 Part two: the data keyword in *PFEM

This part concerns the data in *PFEM. Most of the data are connected with the perturbation of the random variables. The data can be divided into two groups

- a. MV analysis (*MVFO)
- b. AMV analysis (*AMVFO)

2.1 *RESTART

The format of this data keyword is

***RESTART** parameter

where for

parameter = 0 (default): no restart, the NESSUS must generate the database.

parameter = 1: FEM database already exists, bypass FEM database generation.

parameter = 2: MV solution already computed. Perform AMV solution directly.

The parameter will be set to the default value of zero.

2.2 *DATATYPE

The format is

***DATATYPE** parameter

The parameter here specifies the data type to extract

=0 Incremental (dynamic and plastic problem)

=1 Eigenvalue (eigenvalue problem)

=2 Harmonic/spectral (harmonic analysis)

The correct value will be set according to the *PFEM keywords.

2.3 *HARM

The format is

***HARM** parameter1 parameter2

parameter1 = beginning harmonic/spectral number of MV analysis (default = 1)

parameter2 = ending harmonic/spectral number for MV analysis (default = parameter1)

This is only for harmonic analysis and must be input by users.

2.4 *RANVAR

The format is

***RANVAR** parameter (the number of random variables)

Var1 Var2 ... VarN

The default is to use all random variables-1, n.

2.5 *RESPTYPE

The format is

***RESPTYPE** parameter

This keyword defines the response variable to extract from the data base. It must be input by the users. The SHELL will give advice to other users about the choice of the parameter (the physical meaning of the choice).

01	a total displacement component
02	a total strain component
03	a total stress component
11	a plastic strain component
12	a backstress component
13	a creep strain component
14	a thermal strain component
17	a generalized strain component ?
18	a generalized stress component ?
30	the eigenvalue for the mode
31	a modal displacement (eigenvector) component
32	a modal strain component, if available
33	a modal stress component, if available
35	the frequency in radian per time
36	the frequency in cycles per time
51	a real component of the displacement
52	a real component of the strain
53	a real component of the stress
61	an imaginary component of the displacement
62	an imaginary component of the strain
63	an imaginary component of the stress
71	the amplitude of the displacement
72	the amplitude of the strain
73	the amplitude of the stress
81	the phase of the displacement
82	the phase of the strain
83	the phase of the stress
91	the mean square value of the displacement
92	the mean square value of the strain
93	the mean square value of the stress
96	a stress velocity value ?

The possible choices will depend on the type of the problem, i.e., not all the choices will be valid for all types of the problems. For the five different kinds of problems, the following table of knowledge-based options will be used to provide only those options appropriate for the analysis. Descriptions will replace the numbers shown.

STATIC	DYNAMIC	EIGENVALUE	PLASTIC	CREEP	PSD
01	51	30	01	01	91
02	52	31	02	02	92
03	53	32	03	03	93
14	61	33	11	13	
17	62	35	12	17	
18	63	36	17	18	
	71		18		
	72				
	73				
	81				
	82				
	83				

2.6 *INCR

The format is

***INCR** parameter1 parameter2

parameter1 = beginning increment number for MV analysis (default = 0)

parameter2 = ending increment number for MV analysis (default = parameter1)

This is for dynamic and plastic analysis and must be input by users or set to the default value of 0.

2.7 *COMP

This defines which component of the response variable will be the function analyzed by FPI. The form of this data keyword is

***COMP** parameter1 (begin component) parameter2 (end component)

This data must be input by the user. However, SHELL will give the users information about the physical meaning of the component according to the parameter selected in *RESPTYPE keyword.

2.8 *MODE

The format is

***MODE** parameter1 parameter2

parameter1 = beginning mode number for MV analysis (default = 1)
parameter2 = ending mode number for MV analysis (default value will be set to 3 by SHELL)

This is only for eigenvalue analysis and must be input by users.

2.9 *NODE

The format is

***NODE parameter1 parameter2**

parameter1 = beginning node number for MV analysis

parameter2 = ending node number for MV analysis

2.10 *PERT

The format is

***PERT parameter**

Pert1 Pert2 ... PertN

parameter is the number of the perturbations to be extracted and default to all perturbations.

3 Part three: the data keyword in *FPI

A lot of the data in this part will be obtained from the keyword data defined in the previous two data sections.

3.1 *RVNUM

The format is

***RVNUM parameter (the number of random variables)**

DATA BLOCK (with the mean value, standard deviation and distribution)

The number of random variables will be obtained from *RANVAR in *PFEM deck directly. The mean value and standard deviation can be obtained from previous input, and users will be requested to input the distribution of the random variables.

3.2 *GFUNCTION

The format is

***GFUNCTION parameter**

In probabilistic analysis using NESSUS/FEM, the value of the parameter should be set to one or two, depending on the choice of first or second order. The expert opinion of the FPI author is one (first order FPI analysis by data set), so the default will be set to one.

3.3 *DATASETNM

The format is

***DATASETNM** parameter (the number of data sets)

The default value of the parameter will be set to the perturbation number plus one.

3.4 *METHOD

The format is

***METHOD** parameter (the option of method)

For FEM analysis the value of the parameter can be selected by the users as 0 (first order reliability method) or 1 (advanced first order reliability method). The default will be 1.

3.5 *ANALTYP

The format is

***ANALTYP** parameter (the type of analysis)

The value of the parameter can be selected by the users to be 0 (FPI defined P levels), 1 (Z levels) or 2 (P levels) by users; the default will be 0.

3.6 *PRINTOPT

The format is

***PRINTOPT** parameter

The value of the parameter will be 0 (short print out) or 1 (long print out), with 0 as the default.

3.7 *ZLEVELS

The format is

***ZLEVELS** parameter (the number of Z levels)
zlevel1 zlevel2 ... zleveln

Valid only if ANALTYPE is set to 1. These data should be input by users.

3.8 *PLEVELS

The format is

*PLEVELS parameter (the number of P levels)

plevel1 plevel2 ... pleveln

Valid only if ANALTYPE is set to 2. These data should be input by users or set to default value -5, -4, -3, -2, -1, 1, 2, 3, 4, 5.

3.9 *DEFRANV

The format is

*DEFRANV

name1

mean standard-deviation distribution

name2

mean standard-deviation distribution

... ...

namen

mean standard-deviation distribution

All the information here is input by users when the random variables are defined. The system will supply a default distribution and standard deviation to users based on the expertise supplied by Prof. Wirsching.

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